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Interior points algorithms

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- Kiev, Kishenev, Batumi:
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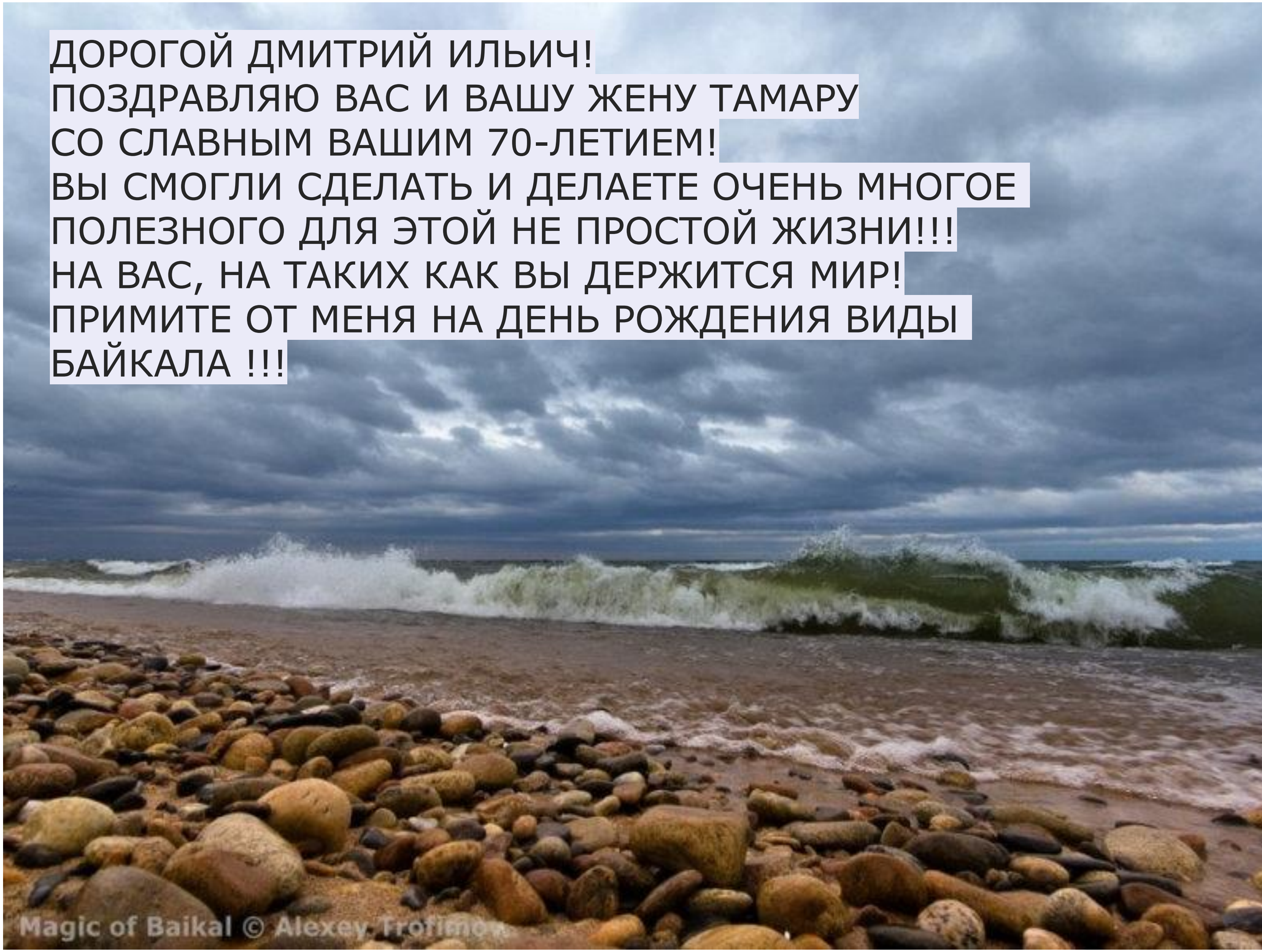
НА ВАС, НА ТАКИХ КАК ВЫ ДЕРЖИТСЯ МИР!
ПРИМИТЕ ОТ МЕНЯ НА ДЕНЬ РОЖДЕНИЯ ВИДЫ
БАЙКАЛА !!!

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The not indisputable definition of an algorithm

This is a way of reducing the problem of finding a solution to a given problem to the solution of a sequence of simpler problems.

It is necessary to distinguish:

1. The ideological basis of the algorithm;
2. The concrete scheme of calculations.

The interior points algorithms discussed below reduce the problem of mathematical programming to the sequential solution of systems of linear equations with a symmetric positively defined matrix. And it is possible to solve not exactly, but with increasing accuracy by iterations, having a good approximation.

Basic ideas of the interior points algorithms

The method of interior points is a family of optimization algorithms that perform improvement of solutions within a domain of vectors satisfying constraint inequalities in strict form.

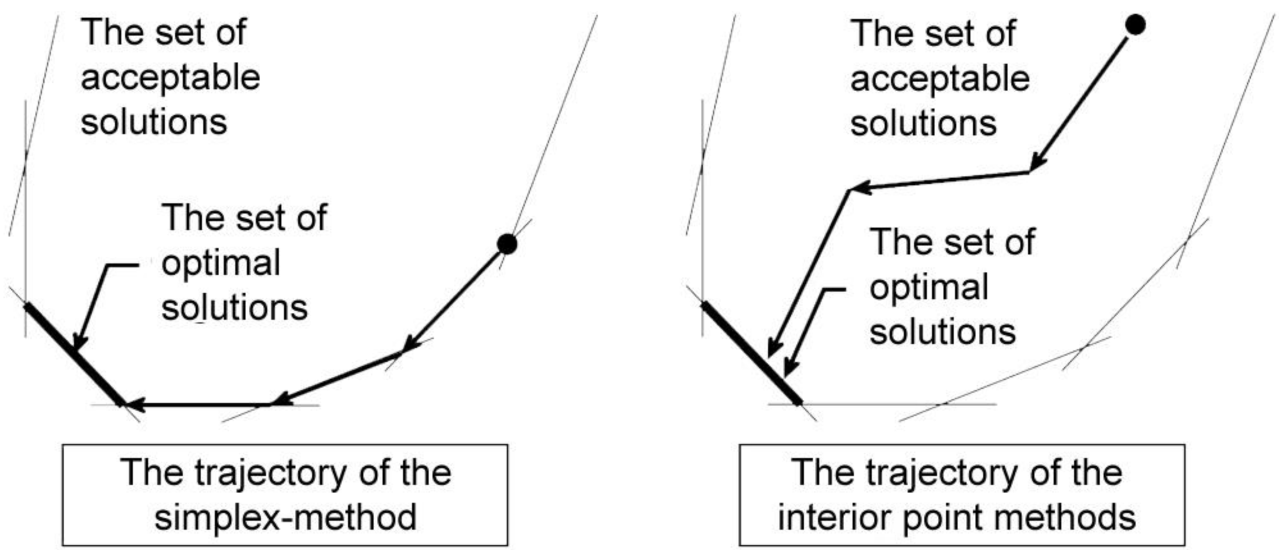
Use special techniques to take into account the degree of approximation to the boundary of this area when choosing the direction of solution improvement:

- Transformations of functions in constraint-inequalities;
- Stimulation of motion from the boundary of the domain;
- Stimulation of motion along the boundary of the domain when approaching it.

They are combined with other methods to account for constraint-inequalities and features of the target function.

Differences of the two types of algorithms for the Linear Programming problem

1. **The simplex method** improves the solution on the polyhedron boundary of vectors satisfying conditions - inequalities. It leads to boundary optimal solutions.
2. **The method of interior points** improves solutions along the trajectories inside the polyhedron. It leads to relatively interior points of optimal solutions, i.e. to solutions with minimal sets of active constraints.



- Симплекс-метод подобен квадратному колесу: на каждой итерации участвует в улучшении решения только часть переменных, одна из них выводится из базиса, другая вводится; крадется по «забору» множества допустимых решений и приходит к крайней точке оптимальных решений, если их несколько;
- метод внутренних точек подобен круглому колесу, на каждой итерации улучшаются решения по всем переменным.
- **Для упрощения изложения будем рассматривать задачу линейного программирования в стандартной форме и двойственную к ней. Реализовывать алгоритмы рекомендую для задач с двусторонними ограничениями на переменные**. Такие задачи имеют преимущества [1]: у них несовместными могут быть только ограничения исходной задачи. У двойственной задачи ограничения заведомо совместные; множество допустимых по ограничениям переменных исходной задачи ограниченное. При этом любой разработчик моделей может (должен уметь) задать диапазон ожидаемых значений переменных.
- *1. Зоркальцев В.И. Решение систем линейных неравенств алгоритмами внутрених точек// Современные методы оптимизации и их приложения к моделям энергетики (сб.науч тр.). – Новосибирск: Наука, 2003. – с. 110-141.*

Mutually dual LP problems

$$c^T x \to \min,\ x \in X, \qquad (1)$$
$$b^T u \to \max,\ u \in U, \qquad (2)$$

where

$$X = \{x \in R^n :\ Ax = b,\ x \geq 0\},$$
$$U = \{u \in R^m :\ g(u) \equiv c - A^T u \geq 0\}$$

- the set of admissible solutions, $c \in R^n,\ b \in R^m,\ A$ - matrix $m \times n$.

The set of optimal solutions of problems (1), (2) we denote by $\overline{X},\ \overline{U}$.

Complementarity non-rigidity conditions: for $(x, u) \in X \times U$

$$(x, u) \in \overline{X} \times \overline{U} \Leftrightarrow x_j g_j(u) = 0,\ j = 1, \ldots, n$$

plays an important role in the interpretation of solutions.

If (1) is a production model, b is a vector of resources, x is a vector of technology intensity, then (2) is a price formation model, u is a vector of resource prices, $-g(u)$ is a vector of technology efficiency

$$-g_j(u) < 0 \Rightarrow x_j = 0,\ x_j > 0 \Rightarrow -g_j(u) = 0.$$

Methodology of L.V. Kantorovich, 1965.

Price formation under a non-optimal plan, based on the minimization of deviations under conditions of complementary non-rigidity.

Let $\widetilde{x} \in X, \ \widetilde{x} > 0$, then

$$\widetilde{u} = arg \min_{u \in R^m} \sum_{j=1}^{n} \widetilde{x}_j (g_j(u))^2.$$

Generalization

$$\widetilde{u} = arg \min_{u \in R^m} \sum_{j=1}^{n} \sigma_j(\widetilde{x}_j)(g_j(u))^2,$$

where σ_j is a monotonically increasing function, $\sigma_j(0) = 0.$

In particular,

$$\sigma_j(\alpha) = \alpha^2.$$

Price Properties:

If $-g_j(\widetilde{u}) > 0$, it is reasonable to increase $\widetilde{x}_j$.

If $-g_j(\widetilde{u}) < 0$, it is reasonable to decrease $\widetilde{x}_j$.

I. I. Dikin's algorithm of interior points, 1967.

Let $x^k \in X,\ x_j^k > 0,\ j = 1, \ldots, n.$

Calculate

$$u^k = arg\min \Phi_k(u), \text{ where } \Phi_k(u) = \sum \left(x_j^k\right)^2 (g_j(u))^2,$$

$$\lambda_k = \sqrt{\Phi_k(u^k)},\ s_j^k = -\left(x_j^k\right)^2 g_j\left(u^k\right),\ j = 1, \ldots, n,$$

$$x^{k+1} = x^k + \lambda_k s^k,\ k = 1, 2, \ldots$$

Dikin's theorem

If $\overline{X} \neq \varnothing$, problem (1) is nondegenerate, then $\exists\ \overline{x} \in ri\overline{X},\ \overline{u} \in ri\overline{U}$ is such that at $k \to \infty$

$$x^k \to \overline{x},\ u^k \to \overline{u}.$$

The rate of convergence is linear.

$ri\overline{X}$ - a subset consisting of the relative interior points of the set $\overline{X}$, a subset of optimal solutions with the maximum set of inactive constraints.

Geometric interpretation of the Dikin method

The vector x^{k+1} resulting from the iterative transition can be represented as the result of solving the minimization problem for a linear function on a convex domain:

$$x^{k+1} = arg\min c^T x$$

under the conditions

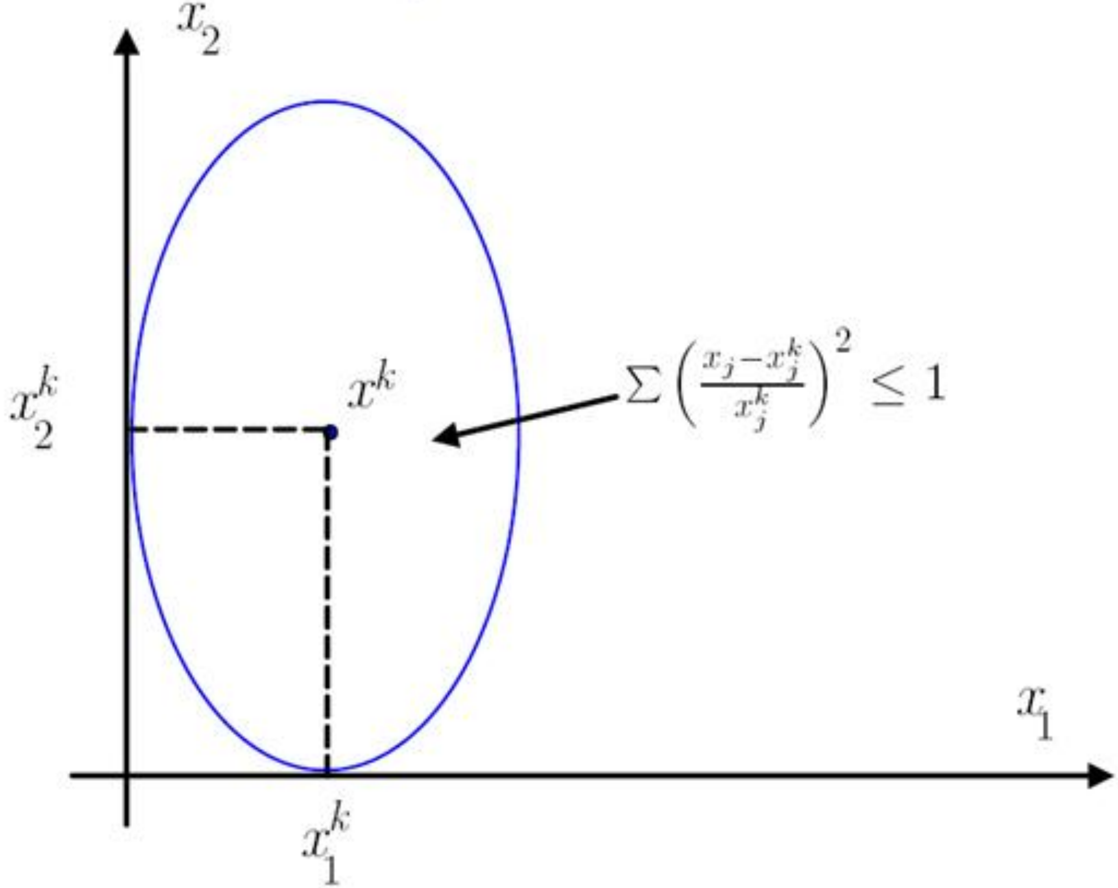
$$A(x - x^k) = 0,$$

$$\sum_{j=1}^{n} \left(\frac{x_j - x_j^k}{x_j^k} \right)^2 \leq 1.$$

Constraint (3) means replacing the non-negativity condition $x \geq 0$ by the condition that x belongs to an ellipsoid inscribed in R_+^n.

Geometric interpretation of the Dikin method

Inscribed in an R^n_+ ellipse



To the history of the method of interior points

1972. I.I. Dikin's thesis defense at the Irkutsk University. Supervisor L.V. Kantorovich, opponents V.L. Makarov, N.Z. Shor.

Interior points algorithms of this type have been actively developed in Russia since the 1970s at the Siberian Energy Institute (Dikin, Zorkaltsev) and at the Computational Center of the USSR Academy of Sciences (Evtushenko, Zhadan). They have been used since the 1970s in models:

1. **Optimization of power systems (EPS) modes;**
2. **Power system reliability analysis;**
3. **Calculation of thermodynamic equilibrium, etc.**

It became very popular after the famous article by N. Karmarkar in 1984, where a worsened modification of I. Dikin's algorithm was presented.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Some publications and results

1. Dikin I.I. Iterative solution of linear and quadratic programming problems - RAS, 1967, No.4.

Complexity of theoretical justification - it is impossible to use Pythagorean theorem or triangle inequality. Fundamentally new fact - convergence to relatively interior point of optimal solutions.

2. Publications of Dikin I.I., Evtushenko Yu.G., Zhadan V.G. on continuous analogues of algorithms.

3. Dikin I.I., Zorkaltsev V.I. Iterative solution of mathematical programming tasks: algorithms of the internal points method. - Novosibirsk: Nauka, 1980.

My results are new rules: choosing a step, setting weight coefficients, entering the area of admissible solutions.

Some publications and results

4. Zorkaltsev V.I. Relative interior point of optimal solutions – Komi SC, 1984. For the first time justification of one (initial) algorithm without the condition of complementary non-rigidity is given.

5. Zorkaltsev V.I. A Family of Interior Point Method Algorithms -Irkutsk, 1985. Dual algorithms, axiomatic approach to representation of set of algorithms, their properties.

6. Zorkaltsev V.I. Justification of family of projective algorithms part 1, part 2 - Irkutsk 1995. Substantiation without nondegeneracy condition for a class of algorithms.

7. Zorkaltsev V.I. Least squares method. - Novosibirsk: Nauka, 1995.

8. Zorkaltsev V.I., Kiseleva M.A. Systems of linear inequalities - Irkutsk: Irkutsk State University, 2007.

The family of interior points algorithms

1. The starting point can be any vector $x^0 > 0,$ **including those that do not satisfy the constraint-inequalities** $Ax = b.$

2. The proposed step selection rule allows to accelerate computational process and to achieve superlinear convergence speed.

3. Axiomatic conditions for choice of weight coefficients allow to use a wide set of specific rules for their setting, including for acceleration of computational process and increasing its stability. Instead of the rule

$$d_j^k = \left(x_j^k\right)^2$$

the general condition is introduced: there are functions $\underline{\sigma},\ \overline{\sigma}$ **such that**

$$\overline{\sigma}(\alpha) \geq \underline{\sigma}(\alpha) > 0,\ \forall\alpha > 0,$$

at which

$$\overline{\sigma}(x_j^k) \geq d_j^k \geq \underline{\sigma}(x_j^k),\ j = 1, \ldots, n.$$

The system of reinforcing requirements for the rules for selecting weighting coefficients

$$\exists\, \overline{\sigma}, \underline{\sigma}: \ \forall \alpha > 0 \ \overline{\sigma}(\alpha) \geq \underline{\sigma}(\alpha) > 0;$$

$$\overline{\sigma}(\alpha) \to 0 \ \text{ at } \ \alpha \to 0;$$

$$\overline{\sigma}\left(x_j^k\right) \geq d_j^k \geq \underline{\sigma}\left(x_j^k\right).$$

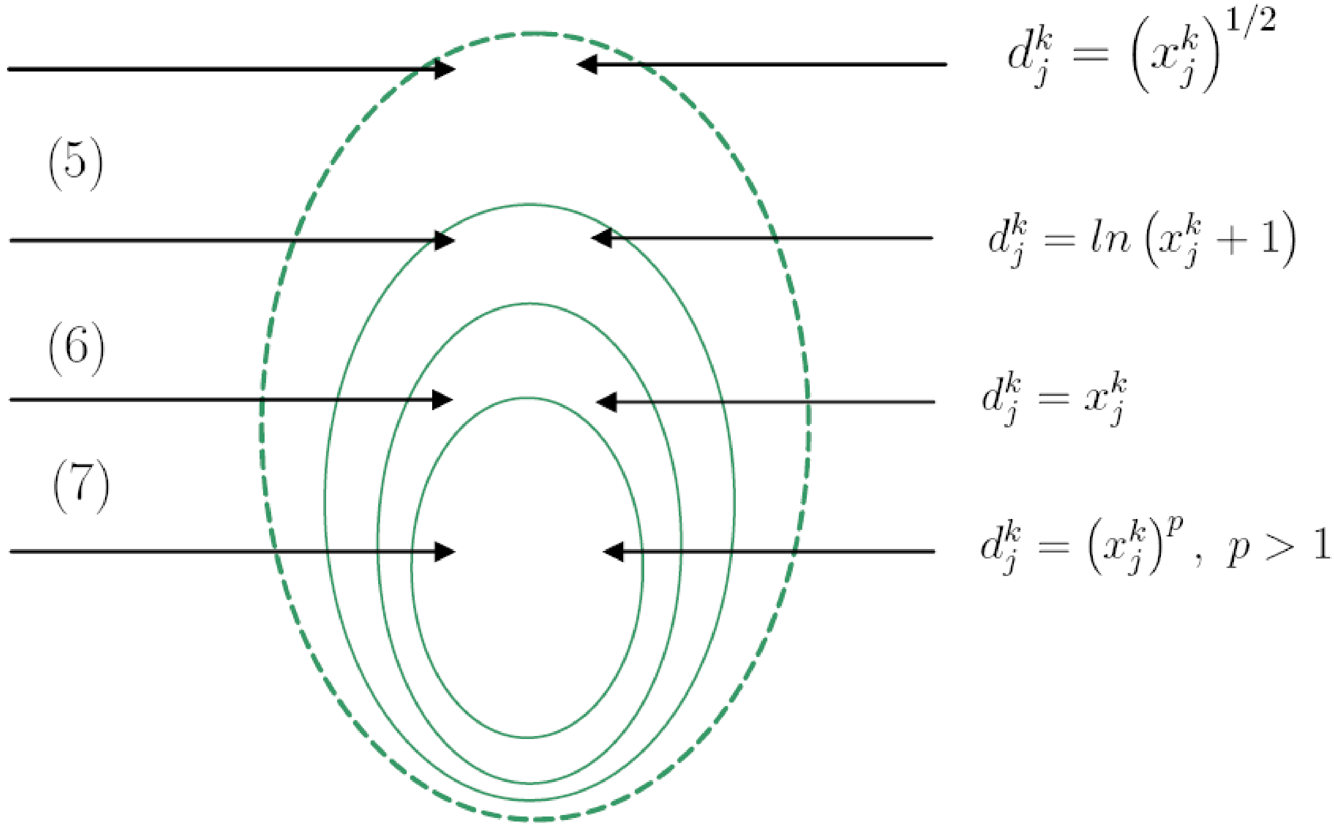
Reinforcing requirements:

$$\overline{\sigma}(\alpha) = O(\alpha), \tag{5}$$

$$\frac{\overline{\sigma}(\alpha)}{\underline{\sigma}(\beta)} = O\left(\frac{\alpha}{\beta}\right), \tag{6}$$

$$\frac{\overline{\sigma}(\alpha)}{\underline{\sigma}(\beta)} = o\left(\frac{\alpha}{\beta}\right). \tag{7}$$

The system of reinforcing requirements for the rules for selecting weighting coefficients



An algorithm that unifying advantages of algorithms with

$$d_j^k = (x_j^k)^2 \text{ and } d_j^k = x_j^k$$

Assume with some $\epsilon > 0$

$$d_j^k = x_j^k / \max\left\{\epsilon,\ \left|g_j(u^{k-1})\right|\right\}.$$

In this case, d_j^k **is not a function only of** x_j^k **(it depends on** u^{k-1} **as well), but there are at which** $\underline{\sigma},\ \overline{\sigma}$**,**

$$\overline{\sigma}\left(x_j^k\right) \geq d_j^k \geq \underline{\sigma}\left(x_j^k\right).$$

The possibility of superlinear convergence rate at $\gamma_k \to 1$ **is proved for the optimization process in the admissible region of such algorithms (under the condition of nondegeneracy). In this case step** λ_k **will be bounded from above. Experiments have confirmed a good convergence speed and stability to errors.**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The family of algorithms

Iterative transition. $x^k \in R^n,\ x^k > 0$ **is set.**

1. Calculate the vector of balance constraint discrepancies $r^k = b - Ax^k.$

2. Supportive problem: Calculate $d^k \in R^n,\ u^k \in R^n$ **from the condition**

$$\overline{\sigma}(x_j^k) \geq d_j^k \geq \underline{\sigma}(x_j^k),\ j = 1, \ldots, n,$$

$$u^k = arg \min_{u \in R^m} \Phi_k(u) - 2(r^k)^T u,$$

where

$$\Phi_k(u) = \sum d_j^k \left(g_j(u)\right)^2,\ g(u) = c - A^T u.$$

3. Find the direction of improvement

$$s_j^k = -d_j^k g_j^k, j = 1, \ldots, n.$$

The family of algorithms

4. Step calculation: at $\gamma \in (0, 1)$

$$\lambda_k = \gamma \cdot \max\left\{\lambda : x^k + \lambda s^k \geq 0\right\},$$

which is equivalent to

$$\lambda_k = \max\left\{\lambda : x^k + \lambda s^k \geq (1-\gamma)x^k\right\},$$

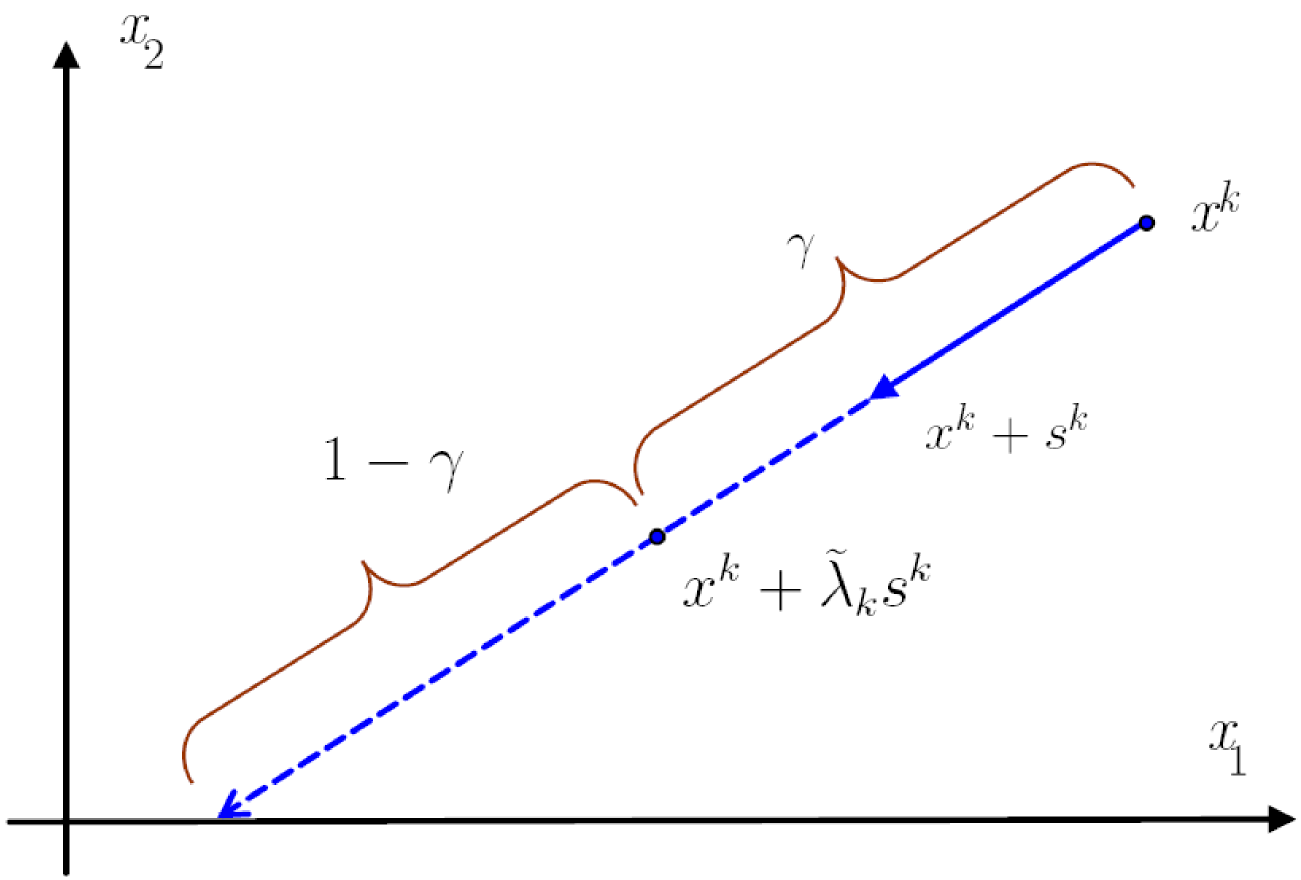
Assume $\lambda_k := \min\{1, \lambda_k\}$ **if** $r^k \neq 0.$

5. Perform an iterative transition

$$x^{k+1} = x^k + \lambda_k s^k.$$

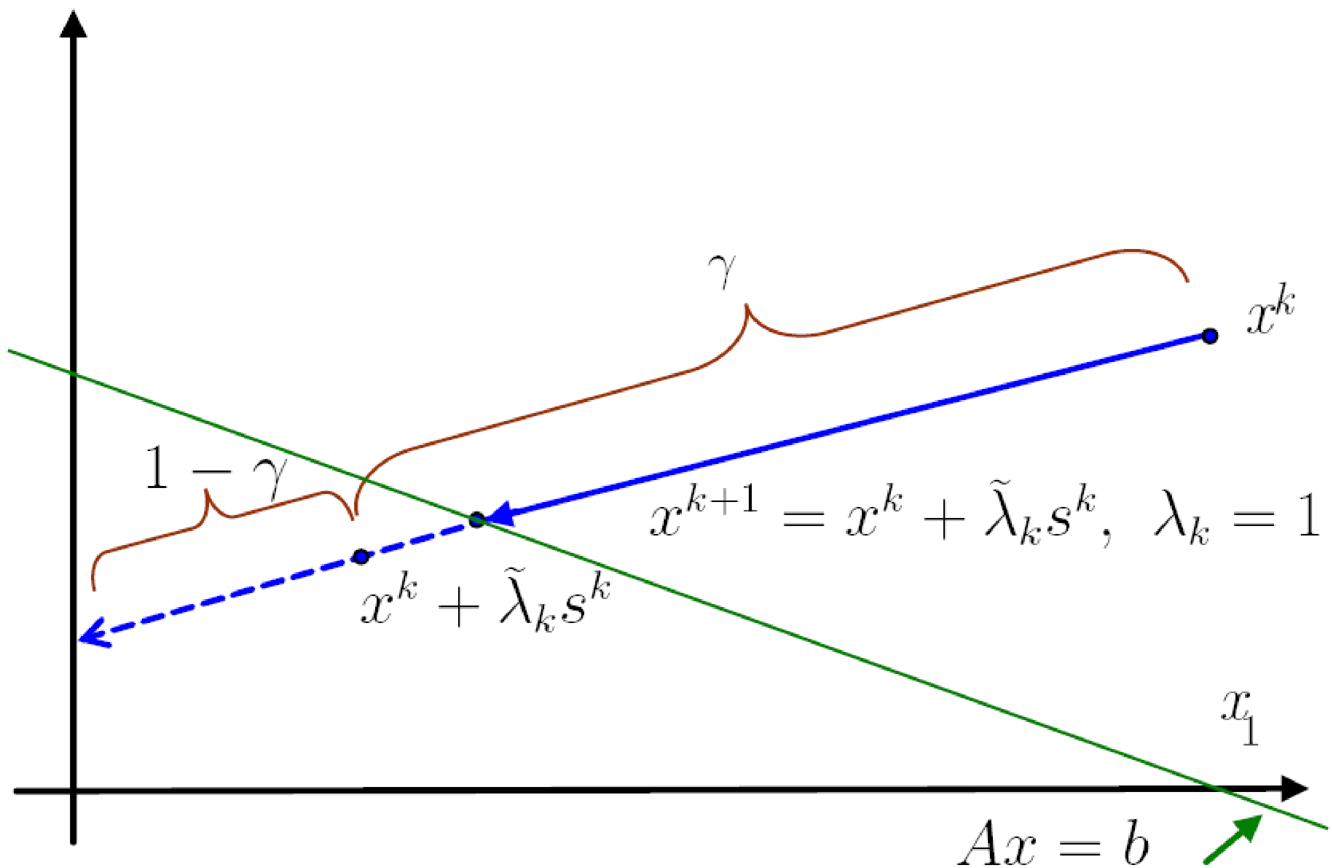
Geometric illustration of the step calculation rule

$$\widetilde{\lambda}_k = \max\left\{\lambda : \; x^k + \lambda s^k \geq (1-\gamma)x^k\right\}$$



Geometric illustration of the step calculation rule

$$\lambda_k = \min\left\{1, \tilde{\lambda}_k\right\} \text{ if } \quad r^k \neq 0.$$



The correction direction

For a given vector of weight coefficients d^k

$$s^k = arg\min\left\{c^T s + \frac{1}{2}\sum (s_j)^2 / d_j^k : As = r^k\right\}.$$

The Lagrange multiples of the constraints compose the vector u^k.

Representing the direction of solution improvement as a sum of two vectors

$$s^k = \hat{s}^k + \check{s}^k,$$

where

$$\check{s}^k = arg\min_{s\in R^n}\left\{\frac{1}{2}\sum (s_j)^2 / d_j^k : As = r^k\right\}$$

- the direction of entry into the region of admissible solutions,

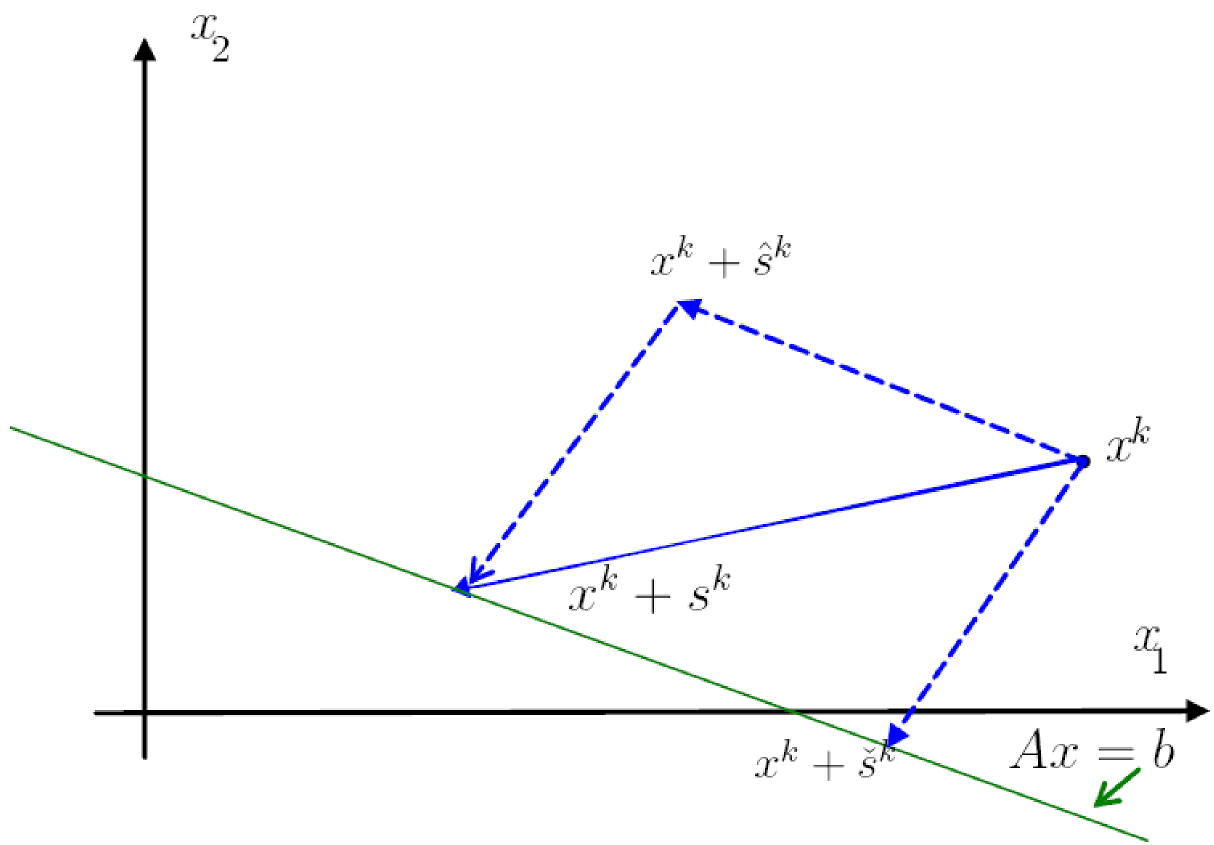
$$\hat{s}^k = arg\min\left\{c^T s + \frac{1}{2}\sum (s_j)^2 / d_j^k : As = 0\right\}$$

- the optimization direction

Geometric representation of the correction direction

$\check{s}$ - **directions of entry into the region of admissible solutions,**

$\hat{s}$ - **optimization direction**.



Variation by iterations of balance constraint discrepancies

The following equality is true

$$r^{k+1} = (1 - \lambda_k) r^k. \qquad (4)$$

Indeed,

$$r^{k+1} = b - Ax^{k+1} = b - A(x^k + \lambda_k s^k) = (b - Ax^k) - \lambda_k A s^k = r^k - \lambda_k r^k.$$

Equality (4) explains why $\lambda_k \le 1$ **at** $r^k \neq 0.$ **Two steps in the calculation are distinguished:**

1. Entering the domain of admissible solutions

If $r^k \neq 0$**, then** $\left\| r^{k+1} \right\| < \left\| r^k \right\|.$

2. Optimization

If $r^k = 0$**, then** $r^{k+1} = 0,\ c^T x^{k+1} < c^T x^k.$

The algorithm is also useful for counteracting accumulated errors in the solution of the supporting problem - to combat "falling out" from the domain of admissible solutions in the process of optimization in it.

Some results for optimization in the admissible domain, $\overline{X} \neq \varnothing$

Theorem. Under the condition of nondegeneracy of the problem:

1. **There are** $\bar{x} \in \overline{X},\ \bar{u} \in ri\,\overline{U}$

$$x^k \to \bar{x},\ u^k \to \bar{u} \quad \text{at} \quad k \to \infty;$$

2. **If (6), then** $\bar{x} \in ri\overline{X}$, **convergence is linear**;
ограничения
3. **If (7), then at** $k \to \infty$

$$\left\|u^{k+1} - \bar{u}\right\| / \left\|x^{k+1} - \bar{x}\right\| \to 0. \qquad \text{(D)}$$

Theorem. For $d_j^k = (x_j^k)^p,\ p \in [1, 3],\ 0 < \gamma \le 2/(1+p):$

1. **Linear convergence of** x^k **and** u^k **to some** $\bar{x} \in ri\,\overline{X},\ \bar{u} \in ri\,\overline{U}$.

2. **At** $p>1$ (D) **is satisfied and** $\left\|x^{k+1} - \bar{x}\right\| / \left\|x^k - \bar{x}\right\| \to (1-\gamma).$

Prospective directions of algorithm development related to the development of new methods of solving the supporting problem, including those taking into account:

1. **Possibility of inaccurate solution at the initial iterations;**

2. **Presence of a good approximation on the subsequent iterations**;

3. **Interval determination of coefficients.**

For example, by using one of the variants of the dual interior points algorithms, the supporting problem is: Find $\vartheta \in R^n,\ d \in R^n$ **satisfying the conditions**

$$\left(A^T A + D\right)\vartheta = f^k,$$

$$\bar{\sigma}(x^k) \geq d_j \geq \underline{\sigma}(x^k),\ j = 1, \ldots, n,$$

where

$$D = diagd.$$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Some kinds of interial point algorithms
(it's time to make a Mendeleev table of them)

1. Affine scaled;
2. Весовые коэффициенты не в виде функций только от переменных;
3. Весовые коэффициенты как искомые величины в заданном интервале;
4. Алгоритмы для задач с двусторонними неравенствами;
6. Central path following;
7. Beveled path following.
8. Расширения …..

1. Straight algorithms;
2. Dual algorithms;
3. Direct-double algorithms that monotonically improve the solution of the self-dual problem:

$$c^T x - b^T u \to \min, \ (x, u) \in X \times U.$$

Comparison of variants of interior point methods for problems of permissible regimes of electric power systems

Algo-rithm	Number of iterations for problems				
	inconsistent		consistent		
	6*7	40*80	2*7	19*3819	201*402
A	1	10	7	23	116
B	1	15	6	24	107
C	1	1	16	13	28
D	1	1	5	5	8
E	1	4	26	24	88

A, B – primal algorithms

E – primal-dual algorithm

C, D – dual algorithms

Facts from the theory of alternative systems of linear inequalities are used to identify the case of no solution.

The theory of unsolvable, (contradictory conditions) **optimization problems was developed in E-burg (Sverdlovsk): S. N. Chernikov, I. I. Eremin, N. N. Astafyev, L. D. Popov, etc. .**

Central path following algorithms

The problem of minimization of the logarithmic barrier function

$$c^T x - b^T u - \mu \sum_{j=1}^{n} \ln(x_j g_j(u)) \to \min_{x \in X,\, u \in U}$$

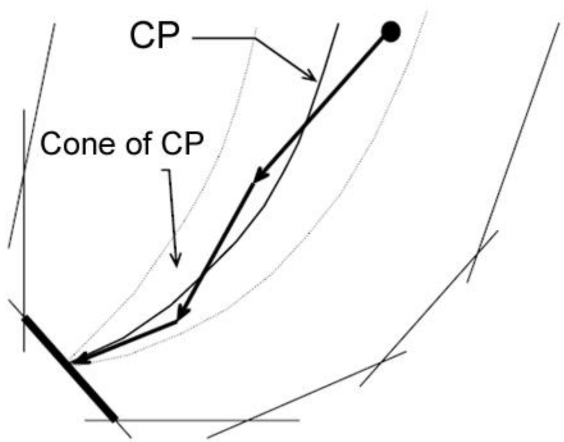
Central path point

$\forall \mu > 0$ there is also a single pair of vectors $x(\mu),\ u(\mu)$:

$$x(\mu) \in X,\ u(\mu) \in U,\ x_j(\mu) g_j(u(\mu)) = \mu,\ \ j = 1,\ldots,n.$$

Center path cone

$$x^k \in X,\ u^k \in U,\ \sum_{j=1}^{n} (\mu^k - x_j^k g_j(u^k))^2 \le \theta(\mu^k)^2.$$



Trajectory of the central path algorithms

Polynomial optimization algorithms in the central path cone

Number of iterations $O(\sqrt{n}L)$

Calculations per iteration

$$O(m^3),\ O(m^{2,5}),\ O(m^2 \ln m).$$

Recalculation rule

$$\mu_{k+1} = (1 - B/\sqrt{n})\mu_k.$$

$B = 0{,}125$ Kojima, Mizuto, and Yoshis, 1989.

$B = 0{,}35$ Monteiro, Adler, 1989.

$B = 0{,}5$ Zorkaltsev 1995, and a very (!) strict inequality

$$\mu_{k+1} < (1 - 0{,}5/\sqrt{n})\mu_k.$$

Beveled path algorithms

Point of beveled path

$\forall t > 0$ **exists and a single pair of vectors** $x(t),\ u(t)$:

$$x(t) \in X,\ u(t) \in U,\ x_j(t) g_j(u(t)) = t_j,\ \ j = 1,\dots,n.$$

The beveled path initiated by vector $t > 0$ **is sets of pairs of vector** $x(\mu t), u(\mu t)$ **at** $\mu > 0$.

Bevel factor

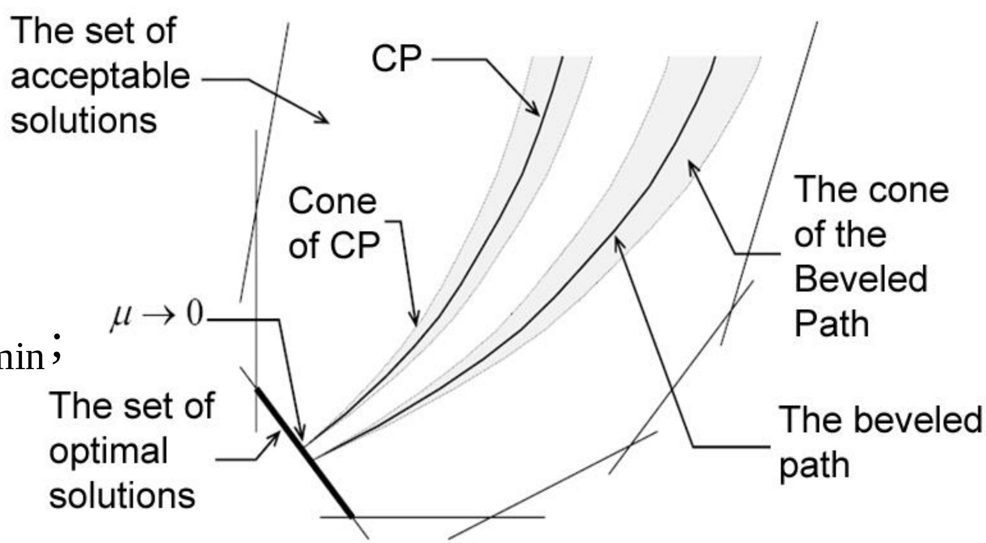
$$\gamma = \bar{t} / t_{\min},\ \gamma \geq 1.$$

Cone of beveled path

$$\Phi_2(x,u,\mu) = \sum_{j=1}^{n} \frac{1}{\mu t_j} (\mu t_j - x_j g_j(u))^2;$$

$$x^k \in X,\ u^k \in U,\ \Phi_2(x^k, u^k, \mu^k) \leq \theta \mu^k t_{\min};$$

$$\mu^{k+1} < \left(1 - \frac{0,5}{\sqrt{n\gamma}}\right) \mu^k.$$



Experimental study (A.Yu. Filatov)

Average number of iterations require to solve random problems

Alg. / Dimensions	**20 x 40**	**50 x 100**	**100 x 200**	**300 x 1000**
Central path	63.6	112.8		
Beveled path	64.6	84.0	98.8	194.0
Initial value μ	688.591	4063.856	7256.781	14601.303
Value μ at iteration 40	1.040	2.008	2.970	21.911

Monte Carlo-based methodology for analyzing the reliability of electric power systems

1. Probabilistic block, in which possible states of electric power system (EPS) are formed randomly.

2. **Block of estimation of power deficit for generated random states (model of minimization of power deficit of electrical power system).**

3. The block of calculating the reliability of the EPS.

A new formulation is given for the power deficit assessment model used in the reliability analysis of electric power systems, which allows combining in a single iterative process:
- entering the domain of admissible solutions,
- minimization of power deficit,
- uniform, load-proportional distribution of the deficit.

Number of iterations of the method of internal points for the implementation of two statements of the model of estimation of the deficit of EPS capacity deficit estimation model.

	Circuit number					
	1	**2**	**3**	**4**	**5**	**6**
Number of nodes	5	5	6	10	11	23
Number of links	5	4	6	9	13	39
Number of iterations in the original formulation	**11**	**35**	**40**	**44**	**45**	**43**
Including:						
entering the domain of admissible solutions	1	2	2	1	1	2
deficit minimization	8	2	7	12	13	10
deficit distribution among the nodes	2	31*	31*	31*	31*	31*
Number of iterations for the realized model in the new formulation	**9**	**7**	**12**	**23**	**19**	**10**

* In these cases, the termination of the calculations occurred not according to the criterion of obtaining the optimal solution, but according to a given number of iterations.

Power Deficit Assessment Model

$$\sum_{i=1}^{n}(\bar{y}_i - y_i) \to \min \quad , \tag{1}$$

$$x_i - y_i + \sum_{j=1}^{n}(1-\alpha_{ji} z_{ji}) z_{ji} - \sum_{j=1}^{n} z_{ij} = 0, \ i = 1, \ldots, n \quad , \tag{2}$$

$$0 \le y_i \le \bar{y}_i, \quad i = 1, \ldots, n \ , \tag{3}$$

$$0 \le x_i \le \bar{x}_i, \quad i = 1, \ldots, n \quad , \tag{4}$$

$$0 \le z_{ij} \le \bar{z}_{ij}, \ \ i = 1, \ldots, n, \quad j = 1, \ldots, n. \tag{5}$$

Calculation results of test circuit #2 (S.M. Perzhabinsky)

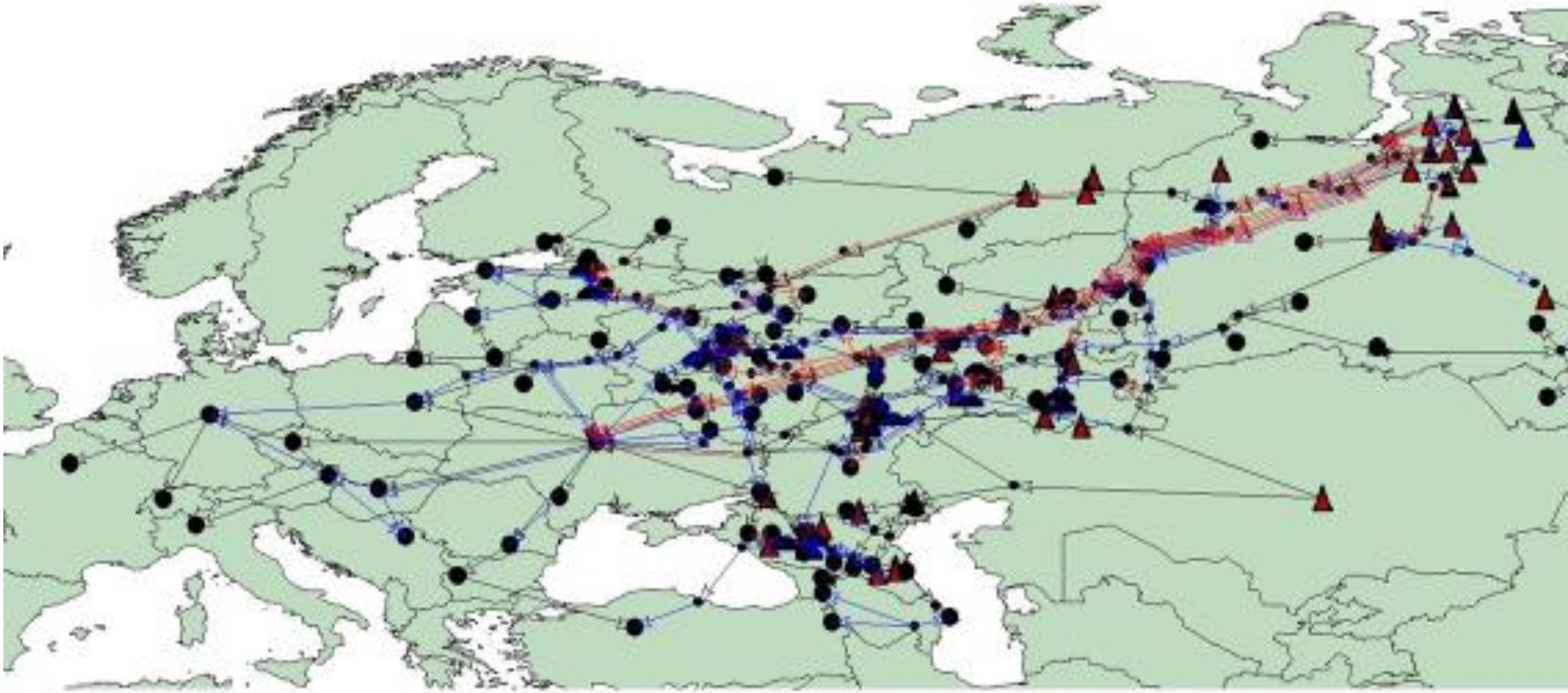
Method	Number of iterations		
	Minimum	**Maximum**	**Average**
Interior points method using quadratic approximations	15	36	22.66
The method of interior points based on linearization	15	87	25.08
Method of interior points with quadratic approximations (balance constraints in the form of equations)	10	23	14.38
Method of interior points based on linearization (balance constraints in the form of equations)	15	55	38.40


Magic of Baikal © Alexey Trofimov

S. Sadov's research on test (obtained from Tallinn) non-good (nonconvex, multi-extremal) problems of interior points algorithms showed their high computational efficiency due to the fact that iteratively produced vector sequences "hover over local extrema".
Zorkaltsev V.I., Sadov S.L. Testing of projective algorithms // Optimization methods and their applications-Irkutsk: SEI SB AS USSR.1989.

An attempt was made to use analogues of interior points algorithms for optimal control problems.

Detailed network for natural gas delivery system



Number of nodes: 337

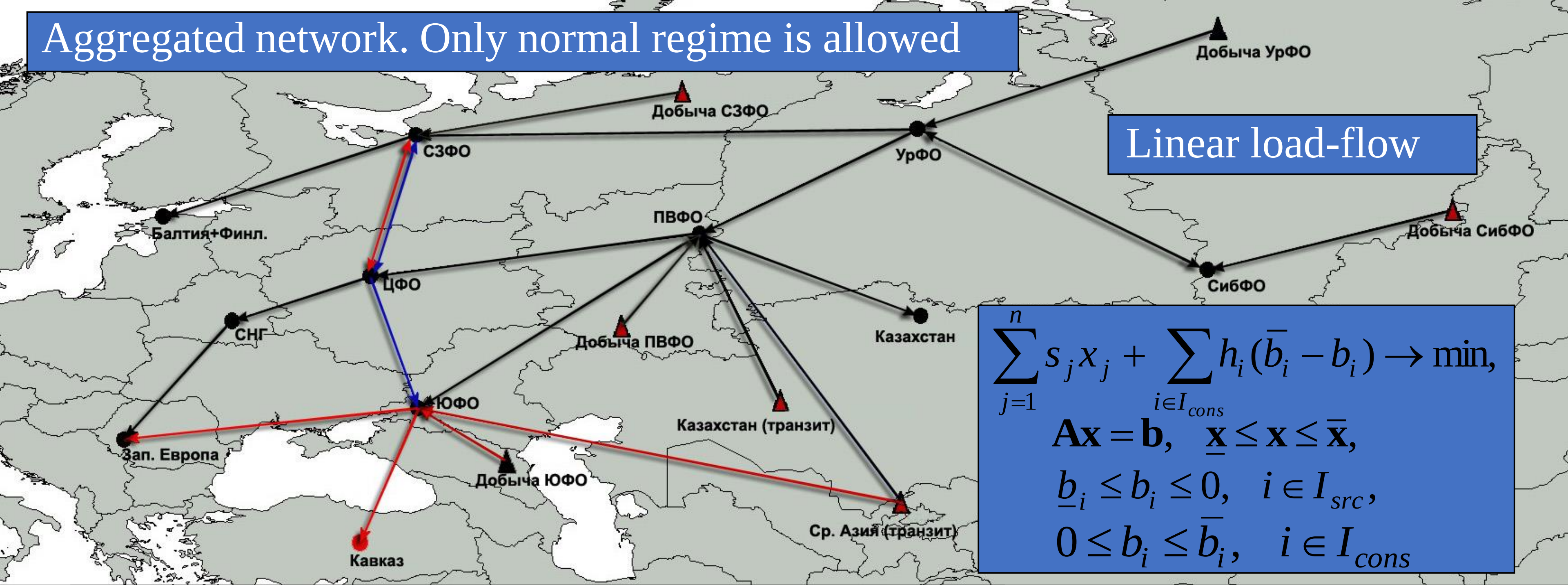
Number of arcs: 589

Amount of iterations of interior points method: 79

Time of calculation: 15.625 sec

60

Aggregated network. Only normal regime is allowed



Linear load-flow

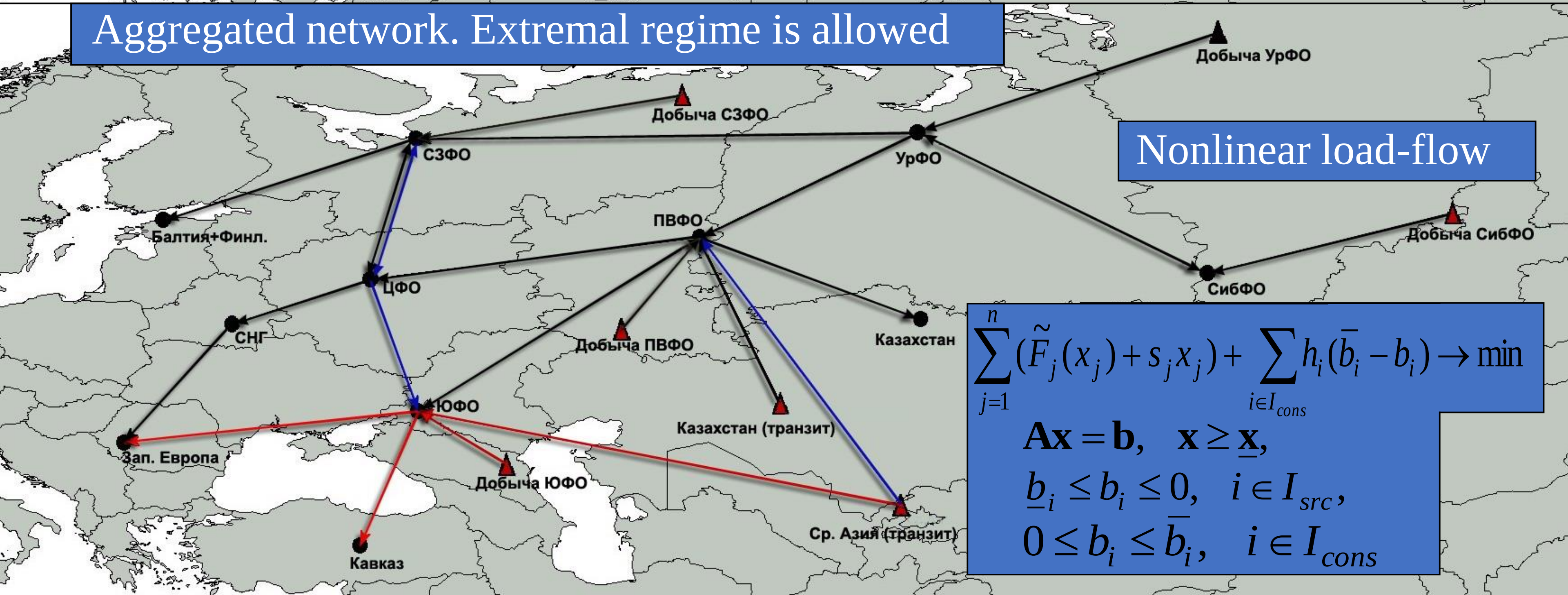
$$\sum_{j=1}^{n} s_j x_j + \sum_{i\in I_{cons}} h_i(\bar{b}_i - b_i) \to \min,$$

$$\mathbf{Ax} = \mathbf{b}, \quad \underline{\mathbf{x}} \le \mathbf{x} \le \bar{\mathbf{x}},$$

$$\underline{b}_i \le b_i \le 0, \quad i \in I_{src},$$

$$0 \le b_i \le \bar{b}_i, \quad i \in I_{cons}$$

Aggregated network. Extremal regime is allowed



Nonlinear load-flow

$$\sum_{j=1}^{n} (\tilde{F}_j(x_j) + s_j x_j) + \sum_{i\in I_{cons}} h_i(\bar{b}_i - b_i) \to \min$$

$$\mathbf{Ax} = \mathbf{b}, \quad \mathbf{x} \ge \underline{\mathbf{x}},$$

$$\underline{b}_i \le b_i \le 0, \quad i \in I_{src},$$

$$0 \le b_i \le \bar{b}_i, \quad i \in I_{cons}$$

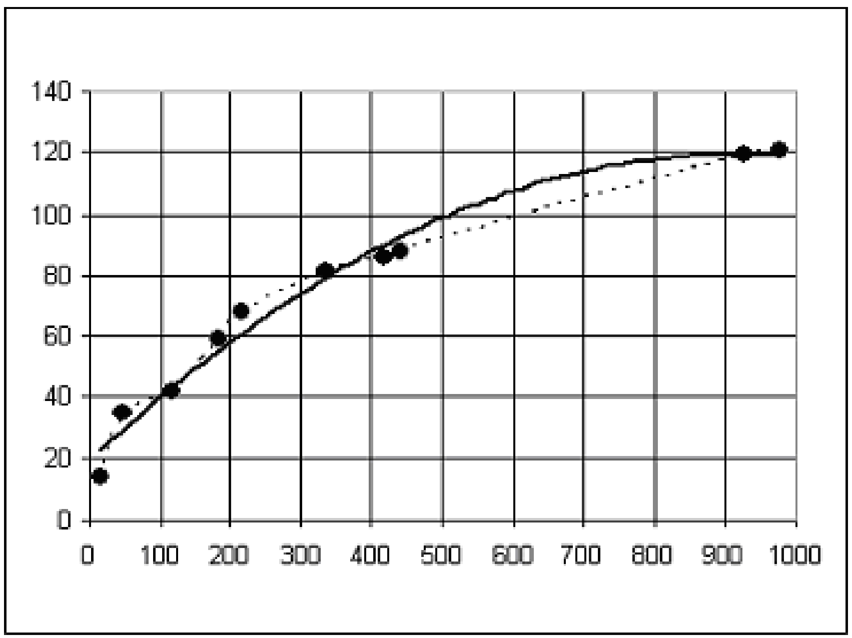
Calculation experiments results (D.S. Medvezhenkov)

Amount of nodes, amount of arcs	Amount of variables in task series	Amount of solved tasks in series	Average number of iterations of interior points method	Average time of calculating of one task in seconds
(7, 10)	17	11	14,00	0,02
(21,28)*	49	15	34,87	0,11
(50, 67)	117	17	42,00	0,44
(75, 109)	184	16	59,44	0,21
(100, 116)	216	16	67,88	0,46
(150,186)	336	16	81,44	1,71
(200, 218)	418	17	85,59	3,57
(200, 240)	440	20	87,80	3,71
(337, 589)*	926	21	119,19	21,26
(360,618)	978	23	121,00	24,73

Diagrams for computation results (D.S. Medvezhenkov)

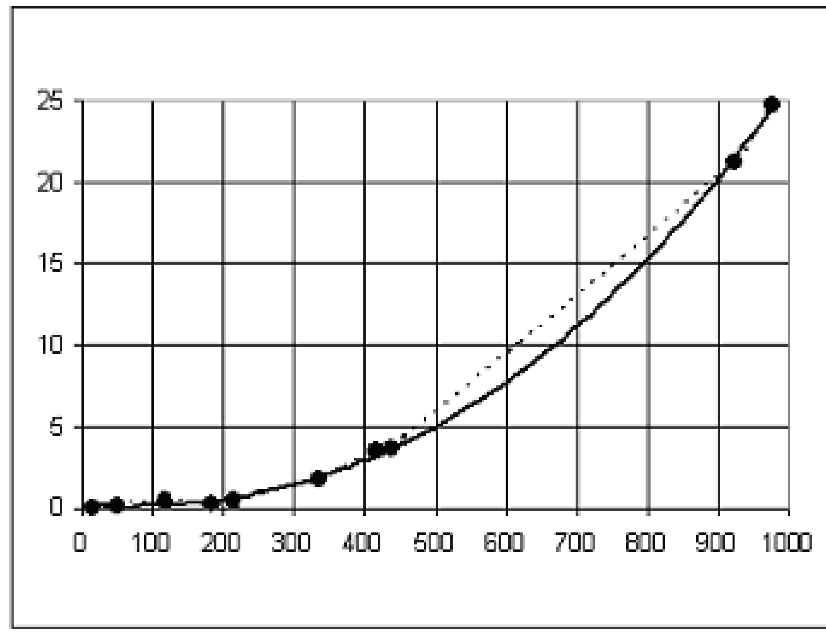
Results of calculations on number of example networks is shown on two diagrams

Number of iterations



Amount of variables

Time of computation, sec



Amount of variables


Magic of Baikal © Alexey Trofimov

Idea for extending algorithms, Frisch's method of potentials, 1955.

Initial Problem:

$$f(x) \to \min,\ x \ge 0,\ x \in X,\ X \subset R^n.$$

Let X be a good set.

The idea of the method:

$$x(t) = \arg\min\{\tilde{f}(x) - t\sum_{j=1}^{n} \ln x_j : x \ge 0,\ x \in X\},$$

Where $\tilde{f}(x)$ a convenient approximation of the f function,
t a positive parameter.

For a wide class of cases

$x(t) \to \bar{x}$ at $t \to 0_+$

where $\bar{x}$ the solution of the original problem.

Real algorithm with a quadratic approximation of the penalty function

Iterative transition

$$x^{k+1} = x^k + \lambda_k s^k,\ k = 1, 2, \ldots,$$

Where $x^k > 0$ the approximation at iteration k,

s^k - direction of correction,

λ_k - the step that provides the reduction of the target function f and conditions $x^{k+1} > 0,\ x^{k+1} \in X$.

Approximation of logarithmic function

$$\ln(x_j^k + s_j) \approx \ln x_j^k + \frac{s_j}{x_j^k} - \frac{1}{2}\frac{(s_j)^2}{(x_j^k)^2}.$$

Real algorithm with a quadratic approximation of the penalty function

The auxiliary problem of finding the direction of solution correction

$$s^k = \arg\min\{ f(x^k + s) - t_k \sum_{j=1}^{n} \frac{s_j}{x_j^k} + \frac{t_k}{2} \sum_{j=1}^{n} \frac{(s_j)^2}{(x_j^k)^2} : x^k + s \in X \},$$

where

$\sum_{j=1}^{n} \frac{s_j}{x_j^k}$ – the component stimulating the motion from the boundaries R_+^n,

$\sum_{j=1}^{n} \frac{(s_j)^2}{(x_j^k)^2}$ – the component stimulating the movement along the boundaries R_+^n.

Modifications of the algorithm

1. It is possible to exclude the component $\sum s_j / x_j^k$. Then there is no need to use the parameter t, (we can put $t = 1$ at all iterations). We obtain Dikin's algorithm.

2. **Leave t only for component** $\sum s_j / x_j^k$.

3. Instead of weight coefficients $d_j^k = \left(x_j^k\right)^2$ other rules of their formation can be used.

In the case of nonlinear constraints when setting X, you can use their linearization (standard way).

Combined algorithm for optimizing the area of admissible solutions of LP problems

The problem

$$f(x) \equiv (c, x) \to \min,\ \ Ax = b,\ \ x \geq 0.$$

Algorithm

$$Ax^k = b,\ \ x^k > 0, k = 1, 2, \ldots$$

$$x^{k+1} = x^k + \lambda_k s^k(\beta^k),$$

where the vector $s^k(\beta)$ at a given parameter β is the solution of problem

$$(c, s) - \beta \sum \frac{s_j}{x_j^k} + \frac{1}{2} \sum \frac{(s_j)^2}{(x_j^k)^2} \to \min,\ \ As = 0.$$

Note that

$$s^k(\beta) = s^k(0) + \beta s^k(1).$$

Experimental study of variants of interior points algorithms (A.Yu. Filatov)

Mean value and standard deviation of the number of iterations required to solve the problems

Algorithms \ Size of tasks	20×40	40×80	100×200	200×500
Affinity-Scaling	$it = 31{,}1$ $\sigma = 6{,}39$	$it = 33{,}0$ $\sigma = 6{,}42$	$it = 29{,}1$ $\sigma = 11{,}62$	$it = 28{,}6$ $\sigma = 2{,}76$
Combined $\beta \in [0,\ 1]$	$it = 25{,}1$ $\sigma = 4{,}44$	$it = 24{,}1$ $\sigma = 4{,}97$	$it = 23{,}1$ $\sigma = 7{,}99$	$it = 22{,}6$ $\sigma = 0{,}66$
Combined Extended $\beta \in [0,\ 2]$	$it = 23{,}6$ $\sigma = 5{,}28$	$it = 21{,}7$ $\sigma = 2{,}87$	$it = 23{,}6$ $\sigma = 8{,}49$	$it = 22{,}5$ $\sigma = 0{,}67$

Experimental study (A.Yu. Filatov)

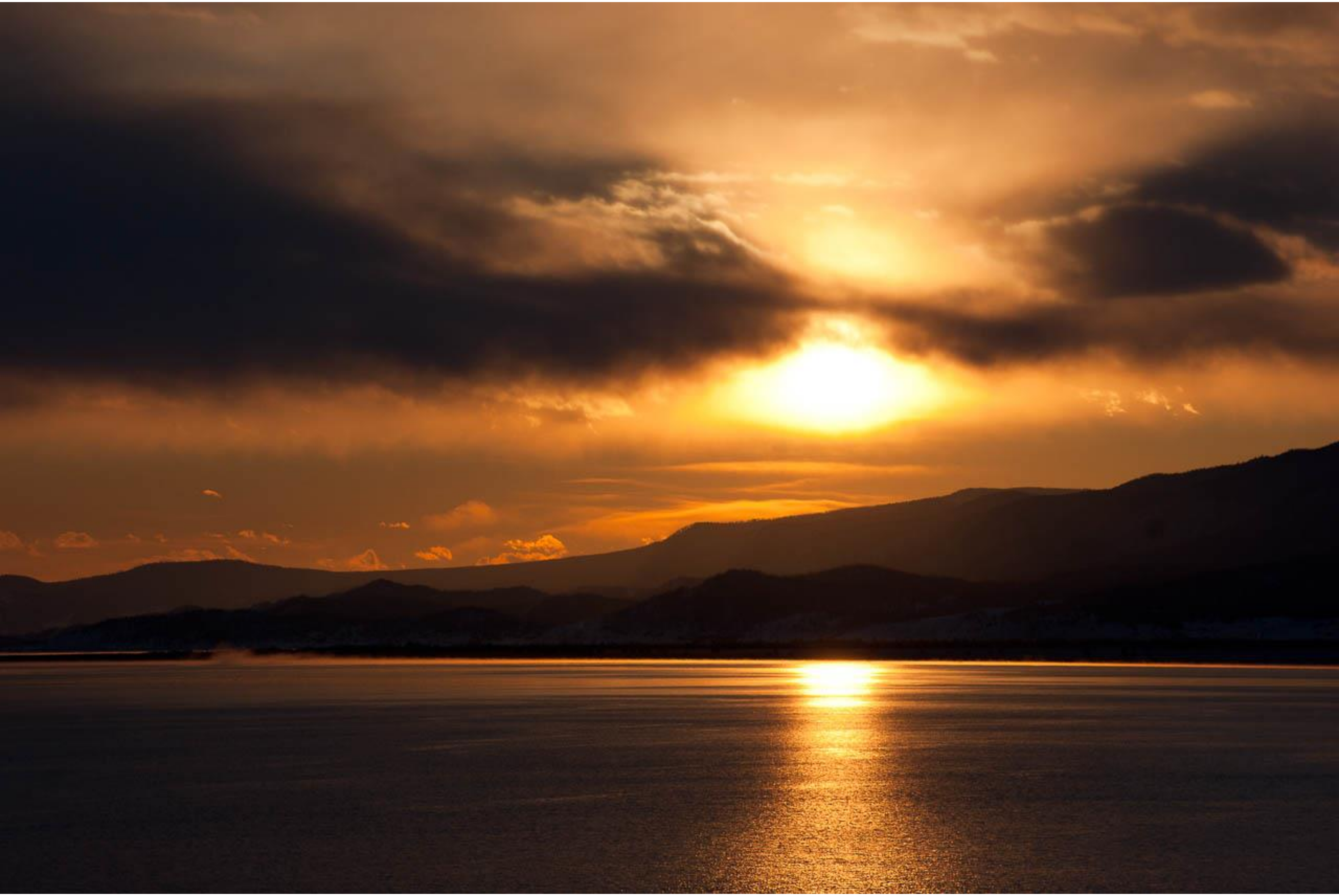
Average, minimum, and maximum number of iterations required to solve joint and incompatible problems.

Algorithms / problems	**Joint problems** (30 x 80) – (41 x 80)	**Non-collaborative problems** (30 x 80) – (41 x 80)
Affine scaling	9.5 (3 - 13)	1.6 (1 - 5)
Combined	5.7 (1 - 8)	1 (all 1)

Studies on the original nonlinear problem

Number of iterations required to solve the initial problem of finding the permissible modes of the EPS

Algorithms / Schemes	Zmin	Ieee118f	Kuzbf	Zimkrasn
Affine-scaled	11 (3 glob.)	54 (11 glob.)	39 (7 glob.)	74 (9 glob.)
Combined	4 (3 glob.)	50 (17 glob.)	33 (3 glob.)	55 (6 glob.)



Thank you for your attention!

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The interior points algorithms are beautiful, ideologically rich, simple and effective in implementation, replaced the simplex method in training courses.

They led to interesting theoretical investigations including optimization problems, nearest to the origin coordinate points of linear manifolds and polyhedras, Chebyshev approximations.

НАЧАЛО ПОЭМЫ «ДМИТРИЙ ИЛЬИЧ СОЛОМОН, 70-ЛЕТИЕ!»

В чем то он чуть-чуть Гордон [1] ,
Сын Давида [2]в чем-то он.
Но с душой неповторимой,
Дорогой Ильич [3]наш, Дима!

1. ***Гордон Д.И.*** *журналист, берущий у всех подряд интервью, знаменит тем, что имеет одинаковое с Д.И. Соломоном имя и отчество.*
2. ***Соломон Давыдович ,*** *хотя и царь, но, в отличии от Д.И.Соломона, так и не смог понять чем надо ему заниматься: то ли разбрасывать камни, то ли их собирать; то ли обнимать женщин, то ли избегать этого?*
3. Не путать с ***Леонидом Ильичем****, из-за которого в истории после допетровского и послепетровского периодов появился днепропетровский период.*