

Two-Stage Transportation Problem Modifications and Their Applications

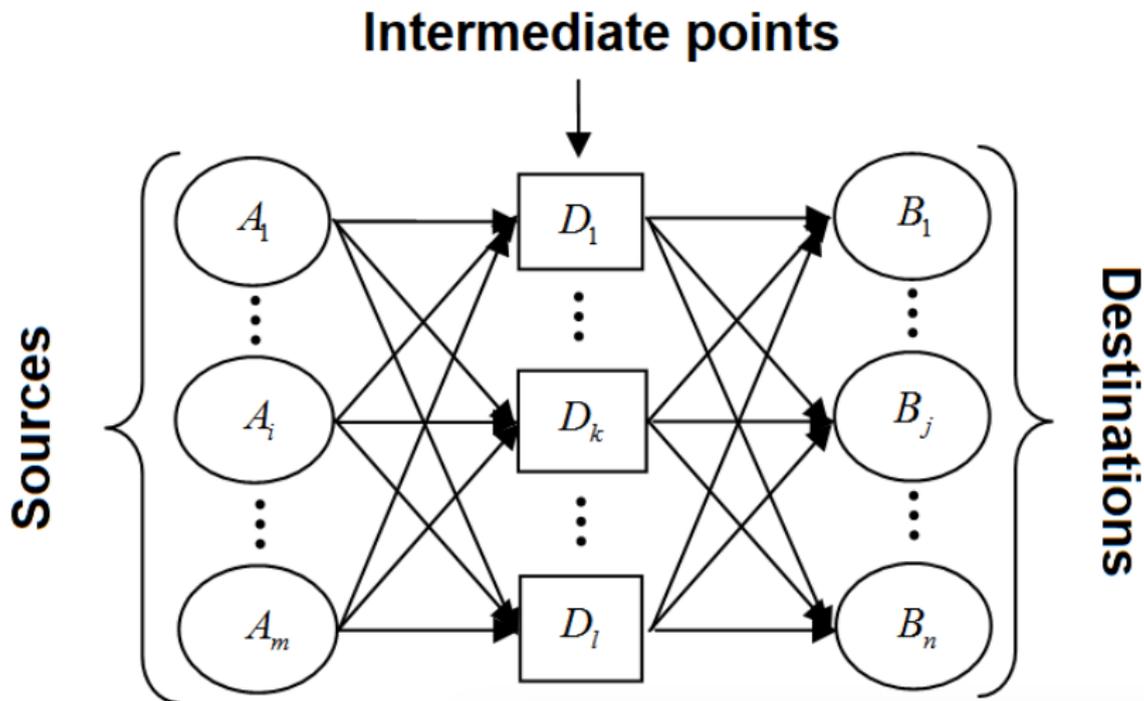
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**On Applications of Gurobi and CPLEX for Solving Two-Stage
Transportation Problem and Its Modifications**

September 6-8, 2022

"A \rightarrow D \rightarrow B" connections



Contents

- 1 Balanced Two-Stage Transportation Problem (BTTP)
- 2 First Modification of BTTP
- 3 Second Modification of BTTP
- 4 Application 1: Optimal Set Partitioning
- 5 Application 2: p -median Problem
- 6 Application 3: Rational Location of n Storages

Contents

- 1 Balanced Two-Stage Transportation Problem (BTTP)
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- 3 Second Modification of BTTP
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Problem

Let A_1, \dots, A_m be m sources containing a_1, \dots, a_m items of goods that must be transported to n destinations B_1, \dots, B_n , which have needs b_1, \dots, b_n . For goods transportation l intermediate points D_1, \dots, D_l could be engaged.

The problem is to find optimal transportation plan, where c_{ik} – unit cost of shipping from source A_i to intermediate point D_k , c_{kj} – unit cost of shipping from intermediate point D_k to destination B_j .

Variables

x_{ik} – the number of items, transported from source A_i to intermediate point D_k

y_{kj} – the number of items, transported from intermediate point D_k to destination B_j

Formulation of the BTTP as LP-problem

$$f_{xy}(x^*, y^*) = \min_{x, y} \left\{ f(x, y) = \sum_{i=1}^m \sum_{k=1}^l c_{ik} x_{ik} + \sum_{k=1}^l \sum_{j=1}^n c_{kj} y_{kj} \right\} \quad (1)$$

$$\sum_{k=1}^l x_{ik} = a_i, \quad i = \overline{1, m}, \quad (2)$$

$$\sum_{k=1}^l y_{kj} = b_j, \quad j = \overline{1, n}, \quad (3)$$

$$\sum_{i=1}^m x_{ik} - \sum_{j=1}^n y_{kj} = 0, \quad k = \overline{1, l}, \quad (4)$$

$$x_{ik} \geq 0, \quad y_{kj} \geq 0, \quad i = \overline{1, m}, \quad j = \overline{1, n}, \quad k = \overline{1, l}. \quad (5)$$

Problem (1)–(5) is a linear programming problem with $(m + n) \times l$ variables and $m + n + l$ constraints

Consistency Conditions

Lemma 1

Constraints system (2)–(5) is consistent if and only if the following condition is satisfied:

$$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j. \quad (6)$$

Contents

- 1 Balanced Two-Stage Transportation Problem (BTTP)
- 2 First Modification of BTTP**
- 3 Second Modification of BTTP
- 4 Application 1: Optimal Set Partitioning
- 5 Application 2: p -median Problem
- 6 Application 3: Rational Location of n Storages

First Modification of BTTP

$$F_{xy}(x^*, y^*) = \min_{x,y} \left\{ F(x, y) = \sum_{i=1}^m \sum_{k=1}^l c_{ik} x_{ik} + \sum_{k=1}^l \sum_{j=1}^n c_{kj} y_{kj} \right\} \quad (7)$$

$$\sum_{k=1}^l x_{ik} = a_i, \quad i = \overline{1, m}, \quad (8)$$

$$\sum_{k=1}^l y_{kj} = b_j, \quad j = \overline{1, n}, \quad (9)$$

$$\sum_{i=1}^m x_{ik} - \sum_{j=1}^n y_{kj} = 0, \quad k = \overline{1, l}, \quad (10)$$

$$\sum_{i=1}^m x_{ik} \leq d_k^{up}, \quad k = \overline{1, l}, \quad (11)$$

$$x_{ik} \geq 0, \quad y_{kj} \geq 0, \quad i = \overline{1, m}, \quad j = \overline{1, n}, \quad k = \overline{1, l}. \quad (12)$$

Consistency Conditions

Problem (7)–(12) is a linear programming problem with $(m + n) \times l$ variables and $m + n + 2l$ constraints

Lemma 2

Constraints system (8)–(12) is consistent if and only if the following condition is satisfied:

$$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j \leq \sum_{k=1}^l d_k^{up}. \quad (13)$$

Contents

- 1 Balanced Two-Stage Transportation Problem (BTTP)
- 2 First Modification of BTTP
- 3 Second Modification of BTTP**
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- 5 Application 2: p -median Problem
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Second Modification of BTTP

$$F_{xyz}(x^*, y^*) = \min_{x, y, z} \left\{ F(x, y) = \sum_{i=1}^m \sum_{k=1}^l c_{ik} x_{ik} + \sum_{k=1}^l \sum_{j=1}^n c_{kj} y_{kj} \right\} \quad (14)$$

$$\sum_{k=1}^l x_{ik} = a_i, \quad i = \overline{1, m}, \quad (15)$$

$$\sum_{k=1}^l y_{kj} = b_j, \quad j = \overline{1, n}, \quad (16)$$

$$\sum_{i=1}^m x_{ik} - \sum_{j=1}^n y_{kj} = 0, \quad k = \overline{1, l}, \quad (17)$$

$$\sum_{i=1}^m x_{ik} \leq d_k^{up} z_k, \quad k = \overline{1, l}, \quad (18)$$

$$\sum_{k=1}^l z_k = d, \quad (19)$$

$$x_{ik} \geq 0, \quad y_{kj} \geq 0, \quad z_k = 0 \vee 1, \quad i = \overline{1, m}, \quad j = \overline{1, n}, \quad k = \overline{1, l}. \quad (20)$$

Second Modification of BTTP

The problem (15)–(20) is a boolean linear programming problem with $(m + n) \times l$ variables and $m + n + 2l + 1$ constraints

Lemma 3

Constraints system (15)–(20) is consistent if and only if the following condition is satisfied:

$$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j \leq \sum_{k=1}^d d_k^{up}, \quad (21)$$

where $d_1^{up} \geq d_2^{up} \geq \dots \geq d_l^{up}$.

Problem equivalence: if $l = d$ and $d_k^{up} \geq \sum_{i=1}^m a_i$, $i = \overline{1, m}$

Contents

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Application 1: optimal set partitioning

There is a source of homogeneous goods with density function $\rho(x) = 1$ continuously distributed in area

$$\Omega = \{x = (x^{(1)}, x^{(2)}) : 0 \leq x^{(1)} \leq 1, 0 \leq x^{(2)} \leq 1\}.$$

Goods are shipped to intermediate points $\tau_i^I = (\tau_i^{I(1)}, \tau_i^{I(2)})$, $i = \overline{1, 5}$ for primary processing or saving. From the points τ_i^I the goods are transported to points of consumption $\tau_j^{II} = (\tau_j^{II(1)}, \tau_j^{II(2)})$, $j = \overline{1, 3}$. The coordinates of the points τ_i^I and τ_j^{II} are known.

The problem:

$$F(\Omega_1, \dots, \Omega_5) = \sum_{i=1}^5 \int_{\Omega_i} c_i^I(x, \tau_i^I) \rho(x) dx + \sum_{i=1}^5 \sum_{j=1}^3 c_{ij}^{II}(\tau_i^I, \tau_j^{II}) \rightarrow \min,$$

$$\sum_{i=1}^5 \int_{\Omega_i} \rho(x) dx = \sum_{j=1}^3 b_j^{II}.$$

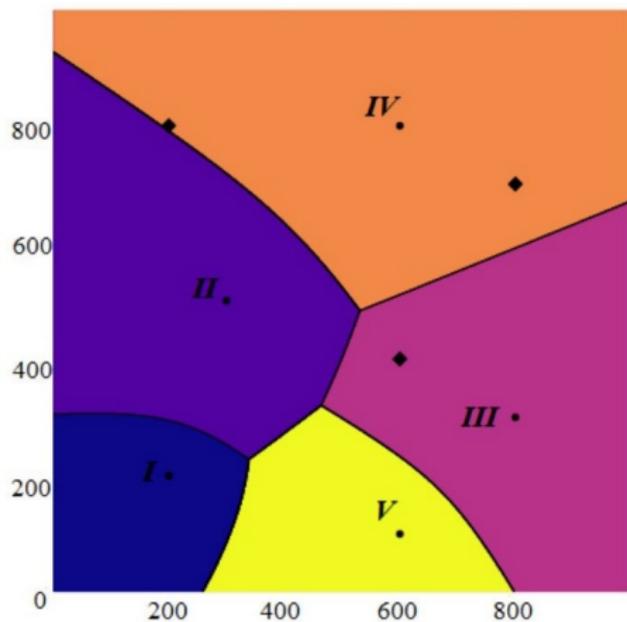
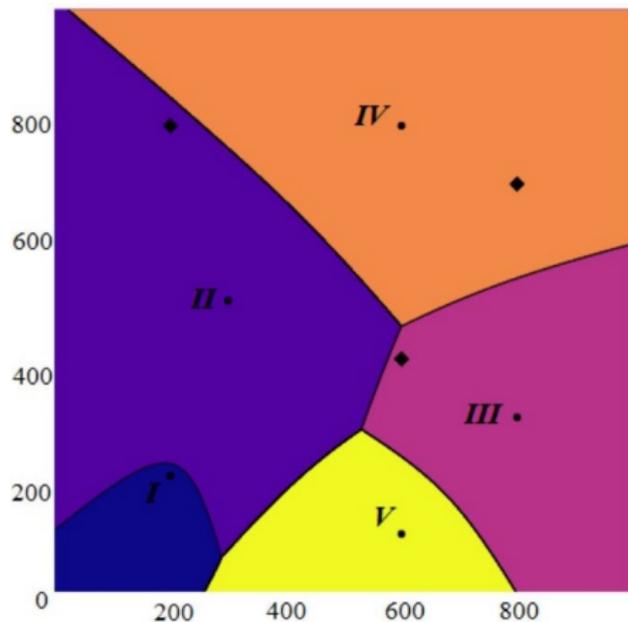
Computational details

Discretization: squares (100×100 , 500×500 , 1000×1000)

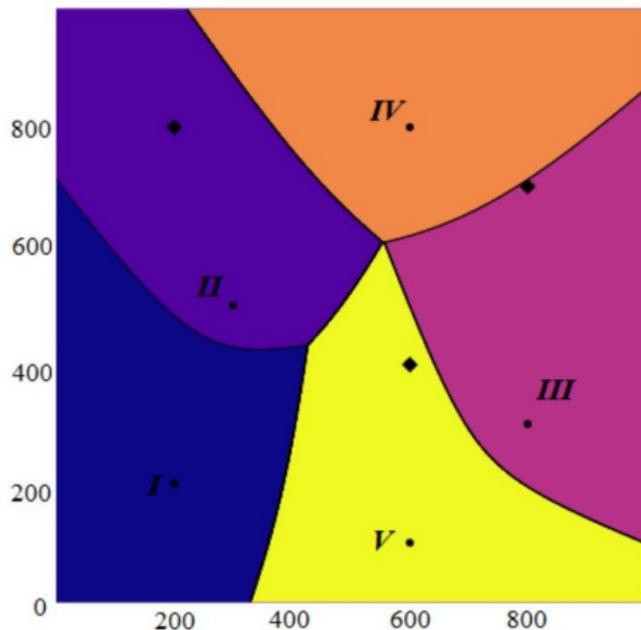
Two Tests

- 1000×1000
 - 1 000 000 sources
 - 5 000 015 variables
 - 1 000 008 linear equality constraints
- CPLEX 20.1.0.0: up to **300 sec avg.**

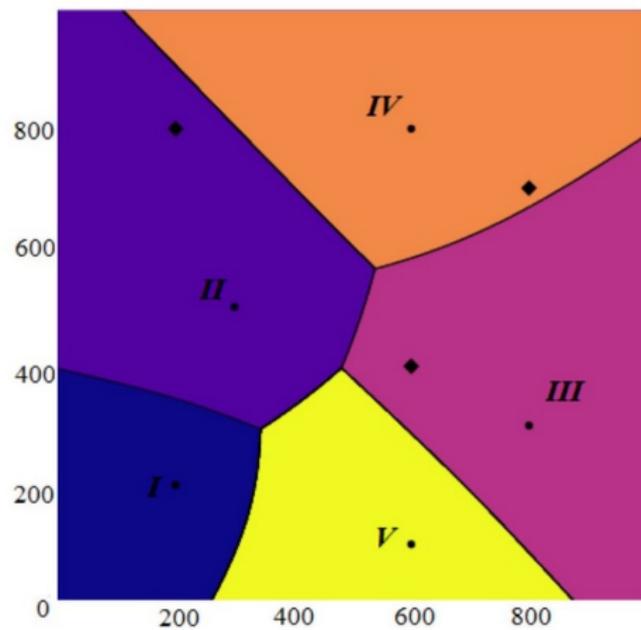
Test 1: two variants of consumer resources



Test 2: two variants of upper bounds



Upper bounds are less than
1/5 of the resource



Upper bounds are less than
1/4 of the resource

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- 1 Balanced Two-Stage Transportation Problem (BTTP)
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Application 2: p-median problem

P-median problem: find nodes x_1, x_2, \dots, x_p in graph G so that the total distance from the nodes $\{x_k\}_{k=1}^p$ to all other nodes is minimal.

If in the problem (14)–(20):

- $l = d$ and $d_k^{up} \geq \sum_{i=1}^m a_i$, $i = \overline{1, m}$;
- sources, intermediate points and consumers are defined via coordinates in \mathbb{R}^2 , identified with nodes in some graph;
- source volumes are equal to 1;
- x_{ik} are boolean and $c_{kj} = 0$,

it is now equivalent to the *p-median problem*.

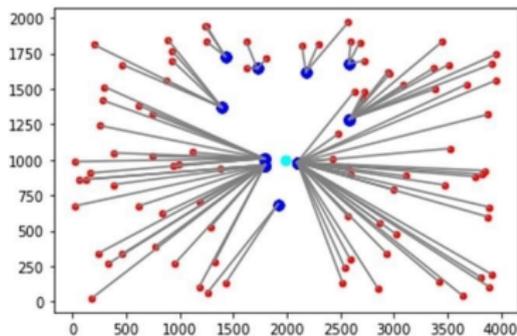
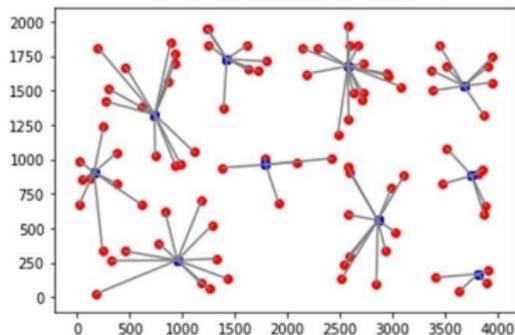
The problem is to choose intermediate points so that the total distance from sources to the intermediate points and from the intermediate points to destinations is minimal.

Computational details

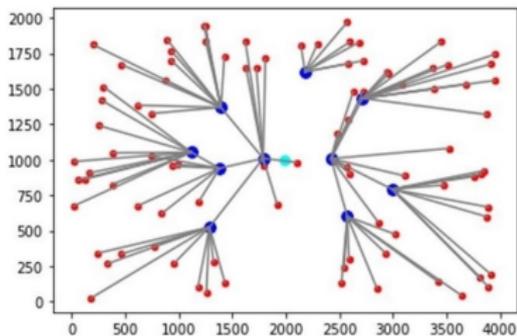
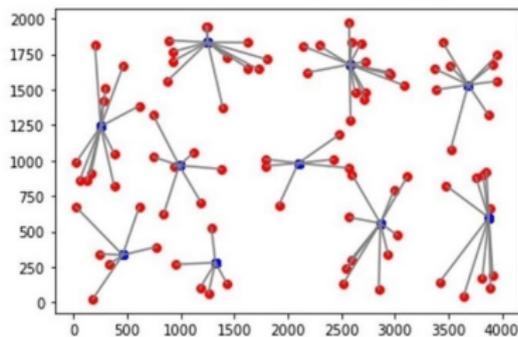
- Data: kroA100.tsp from the library TSPLIB
- Solvers: Gurobi, CPLEX
- **Intermediate points** (blue) are chosen from the **sources** (red)
- **Consumers** are cyan

Results: one consumer

Euclidian distance

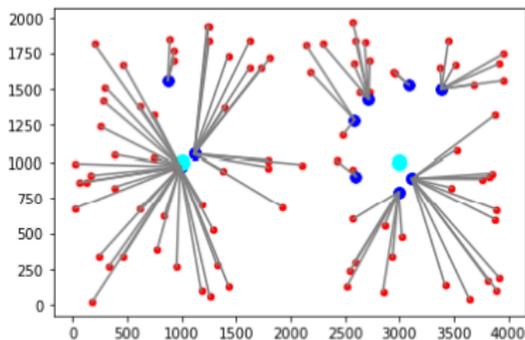
Euclidian distance, $c_{kj} = 0$ 

Euclidian distance squared

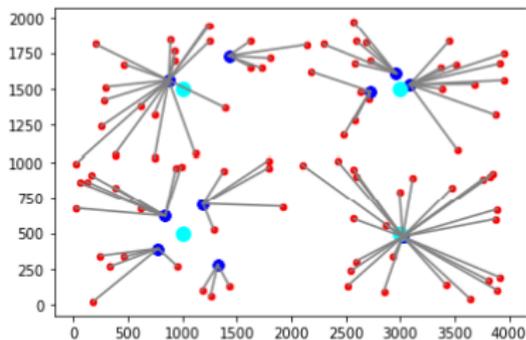
Euclidian distance squared, $c_{kj} = 0$ 

Results: 2-4 consumers

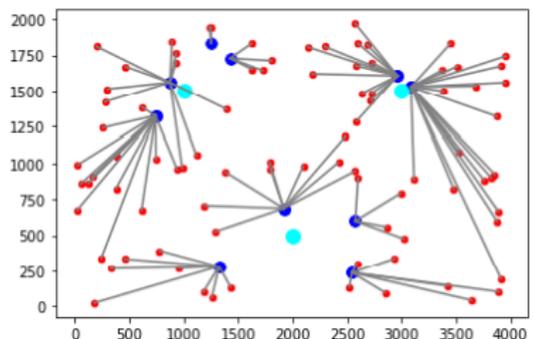
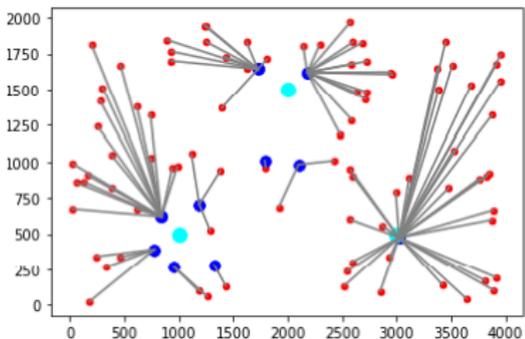
Two consumers



Four consumers



Three consumers



Contents

- 1 Balanced Two-Stage Transportation Problem (BTTP)
- 2 First Modification of BTTP
- 3 Second Modification of BTTP
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Application 3: rational location of n storages

- 100 **sources**, 100 **intermediate points** (10 to be chosen) and 25 **destinations** are defined via coordinates in \mathbb{R}^2 (generated randomly using $U(0, 1)$)
- Resource volumes at sources and destinations are generated randomly using $U(0.1, 2)$

Three problems

- 1 The problem (14)–(20) with linear dependent constraints
- 2 Same, but the last producer constraint is removed
- 3 Same, but the last consumer constraint is removed

Results: $m = 100, n = 100, l = 25, d = 10$

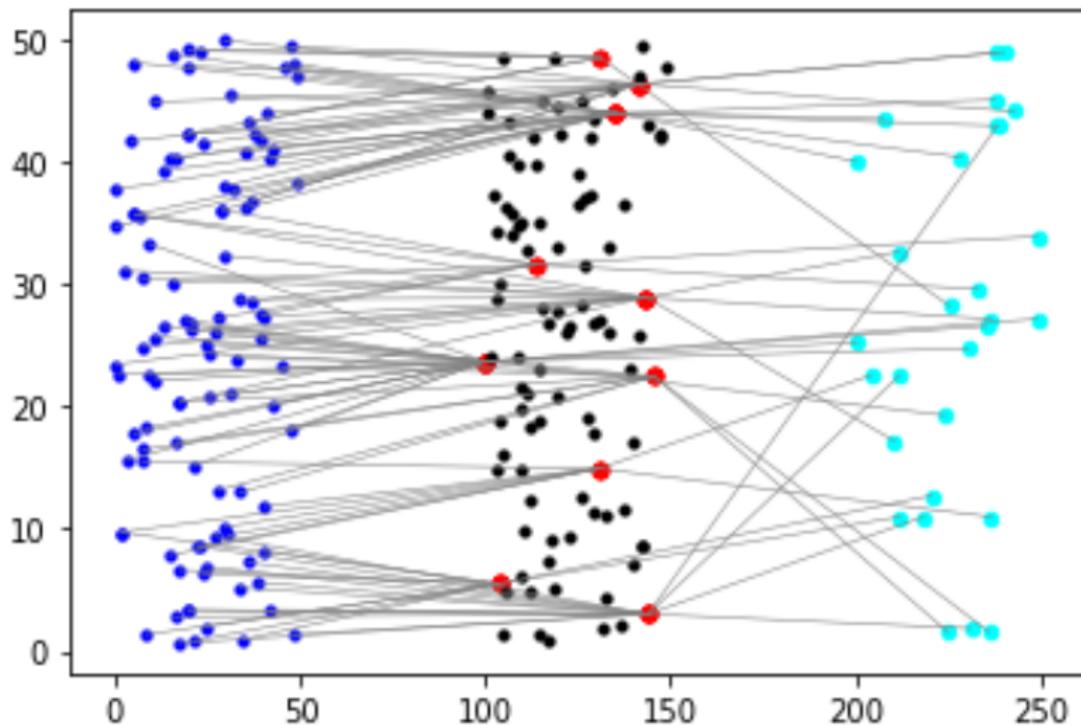


Table 1: CPLEX, $m = 80, n = 80, l = 80$

	Problem 1		Problem 2		Problem 3	
d	f_{xyz}^*	$time, s$	f_{xyz}^*	$time, s$	f_{xyz}^*	$time, s$
2	12898.2	113.94	12898.2	105.79	12898.2	108.65
3	12876.5	1340.94	12876.5	1219.05	12876.5	984.82
4	12869.3	164.96	12869.3	250.28	12869.3	663.05
5	12866.7	78.08	12866.7	80.36	12866.7	73.77
6	12864.8	58.90	12864.8	74.93	12864.8	74.54
7	12863.6	39.75	12863.6	33.54	12863.6	52.64
8	12863.0	32.68	12863.0	21.16	12863.0	16.65
9	12862.7	7.53	12862.7	2.46	12862.7	12.86
10	12862.5	1.13	12862.5	0.73	12862.5	0.72

Table 2: Gurobi, $m = 80, n = 80, l = 80$

d	Problem 1		Problem 2		Problem 3	
	f_{xyz}^*	$time, s$	f_{xyz}^*	$time, s$	f_{xyz}^*	$time, s$
2	12898.4	116.02	12898.2	131.19	12898.2	65.82
3	12877.1	74.80	12876.5	739.15	12876.5	92.76
4	12869.5	60.75	12869.5	86.69	12869.5	55.29
5	12866.6	36.22	12866.6	49.48	12866.6	25.95
6	12865.0	23.14	12864.7	25.43	12864.7	21.69
7	12864.1	22.77	12864.1	24.18	12864.1	13.59
8	12863.5	14.49	12863.3	16.39	12863.1	13.61
9	12863.0	11.70	12862.8	12.77	12862.8	15.02
10	12862.7	2.61	12862.7	2.35	12862.7	2.61

Table 3: CPLEX vs Gurobi, $m = 80, n = 80, l = 80$

d	CPLEX 20.1.0.0			Gurobi 9.1.2		
	$time, s$	N_{iter}	N_{br}	$time, s$	N_{iter}	N_{br}
2	105.79	832860	4234	65.82	119070	1079
3	984.82	4715775	40967	74.80	180774	1417
4	164.96	556308	5345	55.23	135660	1220
5	73.77	289190	2300	25.95	87503	1293
6	58.90	277605	2352	21.69	92711	1762
7	33.54	158922	1329	13.60	47348	1030
8	16.65	60699	661	13.61	60029	1106
9	2.46	3189	0	11.70	48917	1024
10	0.72	1953	0	2.35	4410	15

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Questions?

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