Mathematical models for finding the shortest cycle to visit a given number of vertices from the graph clusters

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- This combinatorial optimization problem has found many applications both in logistics-related areas (vehicle routing, mission planning, etc.), and in more unobvious areas (drilling of printed circuit boards, overhauling gas turbine engines, X-Ray crystallography, etc.)
- However, lots of more complicated routing and sequencing tasks often require many additional constraints to be fulfilled, thus creating a need for improved mathematical models

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Some improvements of TSP

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- Generalized TSP
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- In this work, we present another type of mathematical models

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- Divide all remaining $n - 1$ vertices into K separate clusters $C_1, C_2, ..., C_K$ s.t. $\sum_{k=1}^{K} |C_k| = n - 1$ and $C_p \cap C_q = \emptyset$ for $p \neq q$

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- Denote m_k the number of distinct vertices that must be visited from each cluster $C_k, k \in \overline{1, K}$

Formulation of the problem

- Question: How to find the shortest possible cycle in a graph $D_{n,n}$ that:
  - starts in vertex s,
  - visits all clusters C_k exactly m_k times, without going through the same vertex twice,
  - and then returns to vertex s?

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Basis structure for models A_1 and A_2

$$d^*_{1,2} = \min_{x_{ij}, y_i} \{ d_{1,2} = \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} x_{ij} \} \tag{1}$$

$$\sum_{j=1, j \neq s}^{n} x_{sj} = 1, \sum_{j=1, j \neq s}^{n} x_{js} = 1 \tag{2}$$

$$\sum_{j=1, j \neq i}^{n} x_{ij} = y_i, \sum_{j=1, j \neq i}^{n} x_{ji} = y_i; \; i \in V \setminus \{s\} \tag{3}$$

$$\sum_{i \in C_k} y_i = m_k; \; k \in \overline{1, K} \tag{4}$$

$$(x_{ij}, y_i, u_i) \in B_1 \vee (x_{ij}, y_i, z_{ij}) \in B_2 \tag{5}$$

Model A_1: $(x_{ij}, y_i, u_i) \in B_1$

$$u_i - u_j + Mx_{ij} \leq M - 1;\ i, j \in V \setminus \{s\}, i \neq j \quad (6)$$

$$x_{ij} = 0 \vee 1;\ i, j \in V, i \neq j \quad (7)$$

$$y_i = 0 \vee 1,\ 1 \leq u_i \leq M;\ i \in V \setminus \{s\} \quad (8)$$

Theorem 1

If M is an integer such that $1 \leq M \leq n - 1$, then constraints (2)-(4), (6)-(8) are necessary and sufficient to describe all possible cycles that satisfy the requirements listed above, and minimization of the objective function in (1) corresponds to finding the shortest one among them.

Model A_2: $(x_{ij}, y_i, z_{ij}) \in B_2$

$$z_{ij} - Mx_{ij} \le 0; \;\; i, j \in V, i \ne j \tag{9}$$

$$\sum_{j=1, j\ne s}^{n} z_{sj} = M, \; \sum_{j=1, j\ne s}^{n} z_{js} = 0 \tag{10}$$

$$\sum_{j=1, j\ne i}^{n} z_{ij} - \sum_{j=1, j\ne i}^{n} z_{ji} = -y_i; \;\; i \in V \setminus \{s\} \tag{11}$$

$$x_{ij} = 0 \vee 1, \;\; z_{ij} \ge 0; \;\; i, j \in V, i \ne j \tag{12}$$

$$y_i = 0 \vee 1; \;\; i \in V \setminus \{s\} \tag{13}$$

Theorem 2

If M is an integer such that $1 \le M \le n - 1$, then constraints (2)-(4), (9)-(13) are necessary and sufficient to describe all possible cycles that satisfy the requirements listed above, and minimization of the objective function in (1) corresponds to finding the shortest one among them.

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Clustered st70.tsp

Clusters	Vertices in cluster	To visit
s	53	1
C_1	11, 33, 48, 51, 54, 56, 62, 64, 65, 67	2
C_2	1, 13, 22, 23, 29, 31, 35, 36, 38, 59, 63, 66, 69, 70	3
C_3	2, 3, 7, 8, 19, 26, 28, 32, 49, 55	2
C_4	9, 25, 27, 39, 40, 45, 46, 61	2
C_5	$\{1, \ldots, 70\} \setminus \{s \cup C_1 \cup C_2 \cup C_3 \cup C_4\}$	8

Table 1: Partition of st70.tsp into five mutually exclusive clusters and numbers of vertices to visit from each of them

Clustered st70.tsp

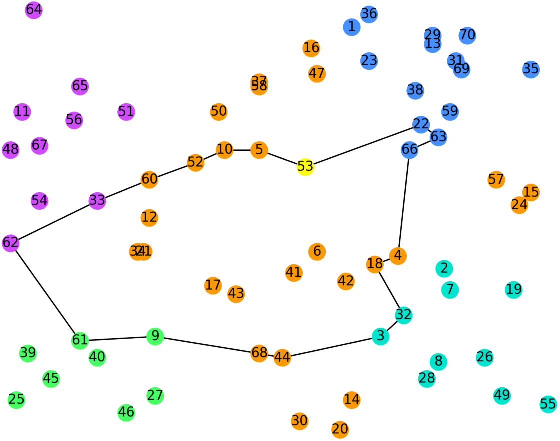

Figure 1: The shortest cycle that fulfills the requirements above

Clustered st70.tsp

Model	min time, sec.	avg time, sec.	max time, sec.
A_1	16907.3	21869.44	30993.8
A_2	152.842	179.813	219.844

Table 2: Results after five runs of AMPL models implementations on the Gurobi 10.0.3 solver for MILP problems from NEOS server

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Conclusion

- Two mathematical models for solving a new type of generalized routing problem were presented
- Necessity and sufficiency of the constraints they are based on for obtaining the exact solution was proved
- An example of solving such problem using Gurobi was provided and performance of two models was compared
- Introduced class of routing problems poses a great interest for future research due to its theoretical value and possibilities it provides in logistics and sequencing applications

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NEOS server
Gurobi solver for MILP problems
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Questions?

Thank You for Your attention!