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МЕТОД ЭЛЛИПСОИДОВ: НЕКОТОРЫЕ ЗАБЛУЖДЕНИЯ

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ТРИ ЗАБЛУЖДЕНИЯ
об Методе Эллипсоидов (МЭ)

1. Кто предложил МЭ и когда?
2. Почему в некоторых источниках утверждается, что МЭ предложил Шор в 1970 году?
3. Почему принято считать, что МЭ не решает задачи с несколькими переменными?

КТО ПРЕДЛОЖИЛ МЭ И КОГДА?

Метод эллипсоидов предложили

1976 **Юдин Д.Б.** и **Немировский А.С.** как метод последовательных отсечений [1].

1977 **Шор Н.З.** как вариант метода с растяжением пространства в направлении субградиента [2].

1. Юдин Д.Б., Немировский А.С. *Информационная сложность и эффективные методы решения выпуклых экстремальных задач* // Экономика и математические методы. – 1976. – Вып. 2. – С. 357–369.

2. Шор Н.З. *Метод отсечения с растяжением пространства для решения задач выпуклого программирования* // Кибернетика. – 1977. – № 1. – С. 94–95.

КТО ПРЕДЛОЖИЛ МЭ И КОГДА?

Chapter 3

The Ellipsoid Method

In 1979 a note of L. G. Khachiyan indicated how an algorithm, the so-called **ellipsoid method**, originally devised for nonlinear nondifferentiable optimization, can be modified in order to check the feasibility of a system of linear inequalities in polynomial time. This result caused great excitement in the world of mathematical programming since it implies the polynomial time solvability of linear programming problems.

This excitement had several causes. First of all, many researchers all over the world had worked hard on the problem of finding a polynomial time algorithm for linear programming for a long time without success. So a really major open problem had been solved.

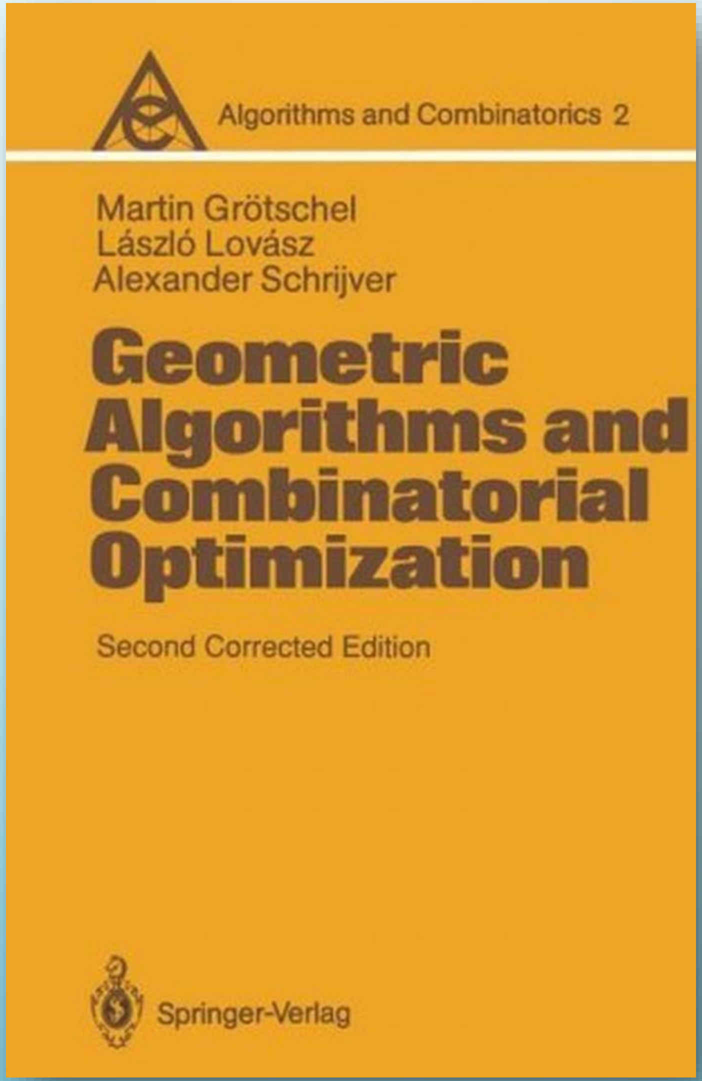
Secondly, many people believe that $\mathcal{P} = \mathcal{NP} \cap \text{co-}\mathcal{NP}$ – cf. Section 1.1 – and the linear programming problem was one of the few problems known to belong to $\mathcal{NP} \cap \text{co-}\mathcal{NP}$ but that had not been shown to be in $\mathcal{P}$. Thus, a further indication for the correctness of this conjecture was obtained.

Thirdly, the ellipsoid method together with the additional number theoretical "tricks" was so different from all the algorithms for linear programming considered so far that the method itself and the correctness proof were a real surprise.

Fourthly, the ellipsoid method, although "theoretically efficient", did not prove to be "practically efficient". Therefore, controversies in complexity theory about the value of polynomiality of algorithms and about how to measure encoding lengths and running times – cf. Chapter 1 – were put into focus.

For almost all presently known versions of the simplex method, there exist a few (artificial) examples for which this algorithm has exponential running time. The first examples of this type have been discovered by Klee and Minty (1972). Such bad examples do not exist for the ellipsoid method. But the ellipsoid method has been observed to be much slower than the simplex algorithm on the average in practical computation. In fact, Borgwardt (1982) has shown that the expected running time of a version of the simplex method is polynomial and much better than the running time of the ellipsoid method. Although the ellipsoid method does not seem to be a breakthrough in applied linear programming, it is of value in nonlinear (in particular nondifferentiable) optimization – see for instance Ecker and Kupferschmid (1983).

As mentioned, nonlinear optimization is one of the roots of the ellipsoid method. The method grew out of work in convex nondifferential optimization (relaxation, subgradient, space dilatation methods, methods of central sections) as well as of studies on computational complexity of convex programming



problems. The history of the ellipsoid method and its antecedents has been covered extensively by Bland, Goldfarb and Todd (1981) and Schrader (1982). Briefly, the development was as follows.

Based on his earlier work, Shor (1970a,b) described a new gradient projection algorithm with space dilatation for convex nondifferential programming. Yudin and Nemirovskiĭ (1976a,b) observed that Shor's algorithm provides an answer to a problem discussed by Levin (1965) and – in a somewhat cryptical way – gave an outline of the ellipsoid method. The first explicit statement of the ellipsoid method, as we know it today, is due to Shor (1977). In the language of nonlinear programming, it can be viewed as a rank-one update algorithm and is quite analogous to a variable metric quasi-Newton method – see Goffin (1984) for such interpretations of the ellipsoid method. This method was adapted by Khachiyan (1979) to state the polynomial time solvability of linear programming. The proofs appeared in Khachiyan (1980). Khachiyan's 1979-paper stimulated a flood of research aiming at accelerating the method and making it more stable for numerical purposes – cf. Bland, Goldfarb and Todd (1981) and Schrader (1982) for surveys. We will not go into the numerical details of these modifications. Our aim is to give more general versions of this algorithm which will enable us to show that the problems discussed in Chapter 2 are equivalent with respect to polynomial time solvability and, by applying these results, to unify various algorithmic approaches to combinatorial optimization. The applicability of the ellipsoid method to combinatorial optimization was discovered independently by Karp and Papadimitriou (1980), Padberg and Rao (1981), and Grötschel, Lovász and Schrijver (1981).

We do not believe that the ellipsoid method will become a true competitor of the simplex algorithm for practical calculations. We do, however, believe that the ellipsoid method has fundamental theoretical power since it is an elegant tool for proving the polynomial time solvability of many geometric and combinatorial optimization problems.

Yamnitski and Levin (1982) gave an algorithm – in the spirit of the ellipsoid method and also based on the research in the Soviet Union mentioned above – in which ellipsoids are replaced by simplices. This algorithm is somewhat slower than the ellipsoid method, but it seems to have the same theoretical applicability.

Khachiyan's achievement received an attention in the nonscientific press that is – to our knowledge – unpreceded in mathematics. Newspapers and journals like The Guardian, Der Spiegel, Nieuwe Rotterdamsche Courant, Népszabadság, The Daily Yomiuri wrote about the "major breakthrough in the solution of real-world problems". The ellipsoid method even jumped on the front page of The New York Times: "A Soviet Discovery Rocks World of Mathematics" (November 7, 1979). Much of the excitement of the journalists was, however, due to exaggerations and misinterpretations – see Lawler (1980) for an account of the treatment of the implications of the ellipsoid method in the public press.

Similar attention has recently been given to the new method of Karmarkar (1984) for linear programming. Karmarkar's algorithm uses an approach different from the ellipsoid method and from the simplex method. Karmarkar's algorithm has a better worst-case running time than the ellipsoid method, and it seems that this method runs as fast or even faster than the simplex algorithm in practice.

ПОЧЕМУ В НЕКОТОРЫХ ИСТОЧНИКАХ УТВЕРДЖАЕТСЯ, ЧТО МЭ ПРЕДЛОЖИЛ ШОР В 1970 ГОДУ?

НОВЫЕ ВАРИАНТЫ МЕТОДОВ ЭЛЛИПСОИДОВ[1]

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Для минимизации выпуклой функции от n переменных предложены алгоритмы ***em80b*** *и* ***em99b****. Алгоритм* ***em80b*** *есть метод эллипсоидов в* B*-форме (Хачиян, 1980), где* $n \times n$*-матрица* B *определяет замену переменных. С его помощью можно найти приближение к точке минимума с очень высокой точностью, чего нельзя сделать с помощью метода эллипсоидов в* H*-форме, где корректируется симметрическая матрица* $H = BB^T$*. Алгоритм* ***em99b*** *является* B*-формой метода эллипсоидов для аппроксимации множества, полученного в результате пересечения* n*-мерного шара и набора определяемых гиперплоскостей (Стецюк, 1999). Его предельный вариант сходится к точке минимума не более чем за n итераций.*

Ключевые слова: *выпуклая функция, метод эллипсоидов, оператор растяжения пространства,* H*- и* B*-формы методов с преобразованием пространства.*

[1] Работа поддержана VolkswagenFoundation (грант № 97775)

ІНСТИТУТ КІБЕРНЕТИКИ
імені В.М. Глушкова НАН УКРАЇНИ

ІНСТИТУТ МАТЕМАТИКИ ТА ІНФОРМАТИКИ
імені Володимира Андрунакієвича МОЛДОВИ
ІНСТИТУТ СИСТЕМ УПРАВЛІННЯ
НАН АЗЕРБАЙДЖАНУ

Матеріали
7–ї міжнародної наукової конференції
МОДЕЛЮВАННЯ І ОПТИМІЗАЦІЯ У ТРАНСПОРТІ ТА ЛОГІСТИЦІ

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5. Stetsyuk P., Fischer A., Khomyak O. The Generalized Ellipsoid Method and Its Implementation. In: Jacimovic M., Khachay M., Malkova V., Posypkin M. (eds) *Optimization and Applications. OPTIMA 2019. Communications in Computer and Information Science.* 2020. Vol 1145. Springer, Cham, pp. 355–370.
6. Grötschel M., Lovász L., Schrijver A. The ellipsoid method and its consequences in combinatorial optimization. Combinatorica. 1981. Vol 1. No 2. pp. 169–197.
7. Стецюк П.И. К методам эллипсоидов. Теория оптимальных решений. Киев: Институт кибернетики им. В.М.Глушкова НАН Украины, 1999. С. 27–33.
8. Шор Н.З. Использование операций растяжения пространства в задачах минимизации выпуклых функций. Кибернетика. 1970. № 1. С. 6–12.
9. Шор Н. З. О скорости сходимости обобщенного градиентного спуска с растяжением пространства // Кибернетика. 1970. № 2. С. 80—85.
10. Стецюк П.И. Методы эллипсоидов и r-алгоритмы. Кишинэу. Эврика. 2014. 488 с.

ПОЧЕМУ В НЕКОТОРЫХ ИСТОЧНИКАХ УТВЕРДЖАЕТСЯ, ЧТО МЭ ШОР ПРЕДЛОЖИЛ В 1970 ГОДУ?

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Published: December 1972 | **6**, 7–15 (1972)

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ПОЧЕМУ ПРИНЯТО СЧИТАТЬ, ЧТО МЭ НЕ РЕШАЕТ ЗАДАЧИ С НЕСКОЛЬКИМИ ПЕРЕМЕННЫМИ

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THE ELLIPSOID METHOD AND COMPUTATIONAL ASPECTS

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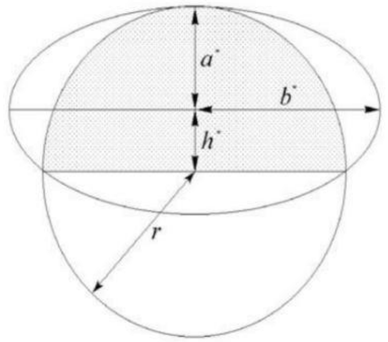
Dedicated to the memory of Professor Naum Z. Shor on the occasion of his 85th birthday

Abstract. This paper gives an overview of particular older and recent results for the ellipsoid method with respect to the contributions by Naum Zuselevich Shor. Therefore, we present this method as a subgradient algorithm with space dilation. For a certain choice of the dilation coefficient, this is a method of outer approximation of semi-ellipsoids by ellipsoids with monotonous decrease in their volume. The paper shows results on properties and applications of the ellipsoid method including computational aspects. Two forms of the ellipsoid method are described which differ in the way of updating the inverse space transformation matrix. The applicability of the ellipsoid method to several problem classes, like convex programs and saddle point problems for convex-concave functions, is discussed. Finally, the acceleration of the ellipsoid method by deeper ellipsoid approximations is also dealt with.

Keywords. Ellipsoid method; Nonsmooth optimization; Subgradient algorithm; Space dilation; Saddle point problem.

2020 Mathematics Subject Classification. 65K05, 90C25, 90C30, 90C90.



$$a^* = r\frac{n}{n+1}$$

$$b^* = r\frac{n}{\sqrt{n^2-1}}$$

$$h^* = r\frac{1}{n+1}$$

FIGURE 1. The parameters of minimal volume ellipsoid containing a half-ball in $\mathbb{R}^n$.

For $\alpha = \frac{b}{a} = \sqrt{\frac{n+1}{n-1}}$, it follows that $q_n(\alpha) < 1$. To transform the ellipsoid, containing a half-ball, into a new ball, it is sufficient to dilate the space of variables in the direction of the minor semi-axis with the coefficient $\alpha = \frac{b}{a}$. This can be done using the operator of space dilation $R_\alpha(\xi)$, where the direction ξ coincides with the direction of the minor semi-axis of the ellipsoid.

If $X = \mathbb{R}^n$ is the original space of variables, then in the transformed space of variables $Y = R_\alpha(\xi)X$, we get a new ball of radius b, which contains the solution of our problem. Repeating this procedure, but for the new ball in the transformed space, we obtain Algorithm 1. Here, in Step 2, the direction of the minor semi-axis of the ellipsoid in the transformed space $Y_k = B_k^{-1}X$ is calculated and the transition to its center is performed. The calculated direction is used for the next space dilation, which is implemented in Step 3 by determining the matrix B_{k+1}. In the next transformed space $Y_{k+1} = B_{k+1}^{-1}X$, we get a ball of radius r_{k+1}.

ПОЧЕМУ ПРИНЯТО СЧИТАТЬ, ЧТО МЭ НЕ РЕШАЕТ ЗАДАЧИ С НЕСКОЛЬКИМИ ПЕРЕМЕННЫМИ

2.2. **The H-form of the ellipsoid method.** The B-form of the ellipsoid method (Algorithm 1) can be written in H-form by means of positive definite symmetric matrices H_k. This is presented and discussed below.

Algorithm 2 – The H-form of the ellipsoid method

Step 0. Choose $x_0 \in \mathbb{R}^n$, a positive definite symmetric matrix $H_0 \in \mathbb{R}^{n\times n}$, and $r_0 > 0$ so that

$$(x_0 - x^*)^\top H_0^{-1}(x_0 - x^*) \le r_0^2.$$

Moreover, set $k := 0$.

Step 1. If $g(x_k) = 0$, then set $x^* := x_k$ and STOP.

Step 2. Calculate

$$x_{k+1} := x_k - h_k \frac{H_k g(x_k)}{\sqrt{g(x_k)^\top H_k g(x_k)}}, \quad \text{where } h_k := \frac{1}{n+1} r_k.$$

Step 3. Update

$$H_{k+1} := H_k - \frac{2}{n+1}\frac{H_k g(x_k) g(x_k)^\top H_k}{g(x_k)^\top H_k g(x_k)} \quad \text{and} \quad r_{k+1} := \frac{n}{\sqrt{n^2-1}} r_k.$$

Step 4. Set $k := k+1$ and go to Step 1.

Algorithm 3 – Algorithm emshor

Step 0. Choose $x_0 \in \mathbb{R}^n$ and $r_0 > 0$ so that $\|x_0 - x^*\| \le r_0$.
Moreover, choose $\varepsilon > 0$, set $B_0 := I_n$ and $k := 0$.

Step 1. If $\left\|B_k^\top g_f(x_k)\right\| r_k \le \varepsilon$, then set $k^* := k$, $x_\varepsilon^* := x_k$ and STOP.

Step 2. Calculate

$$x_{k+1} := x_k - h_k B_k \xi_k, \quad \text{where} \quad \xi_k := \frac{B_k^\top g_f(x_k)}{\left\|B_k^\top g_f(x_k)\right\|}, \quad h_k := \frac{1}{n+1} r_k.$$

Step 3. Update

$$B_{k+1} := B_k + \left(\sqrt{\frac{n-1}{n+1}} - 1\right)(B_k \xi_k)\,\xi_k^\top \quad \text{and} \quad r_{k+1} := \frac{n}{\sqrt{n^2-1}} r_k.$$

Step 4. Set $k := k+1$ and go to Step 1.

The next theorem follows directly from Theorem 2.1, if one substitutes α according to its definition and by taking into account the stopping criterion in Step 1 of Algorithm 3 with the explanation above.

Theorem 3.1. *Let x_k and x_{k+1} be generated by Algorithm 3. Then, the ratio of volumes of the ellipsoids $\mathscr{E}_k$ and $\mathscr{E}_{k+1}$ does not depend on k and is equal to*

$$q_n := \frac{vol(\mathscr{E}_{k+1})}{vol(\mathscr{E}_k)} = \frac{n}{n+1}\left(\frac{n}{\sqrt{n^2-1}}\right)^{n-1} < \exp\left\{-\frac{1}{2n}\right\} < 1.$$

Moreover, $x^* \in \mathscr{E}_k$ holds for all $k = 0, 1, \ldots, k^*$ and, if Algorithm 3 stops, $f(x_\varepsilon^*) \le f^* + \varepsilon$ is valid.

ПОЧЕМУ ПРИНЯТО СЧИТАТЬ, ЧТО МЭ НЕ РЕШАЕТ ЗАДАЧИ С НЕСКОЛЬКИМИ ПЕРЕМЕННЫМИ

3.1. **Octave implementation of Algorithm emshor.** Algorithm 3 (Algorithm **emshor**) was implemented in Octave [4, 13]. It uses a function of the form `function[f,g]=calcfg(x)`, which calculates the value $f(x)$ and a subgradient $g_f(x)$ at x. This function has to be provided by the user. The code of the implementation and some short comments are given below.

Octave code for Algorithm emshor

```
# Input parameters:
# calcfg - name of the function for calculation of f and g
# x0 - starting point, x0(1:n)
# r0 - the radius of the ball localizing the minimum point
# epsf, maxitn - stop parameters (accuracy, max. iter.)
# intp - printing interval (after each intp iterations)
# Output parameters:
# x - approximation of the minimum point, x(1:n)
# f - value of function f at the point x
# itn - number of iterations performed
# ist - exit code (1=epsf, 4=maxitn)
function[x,f,itn,ist]=emshor(calcfg,x0,r0,epsf,maxitn,intp);     #row01
n=length(x0); x=x0; B=eye(n); r=r0;                              #row02
dn=double(n); beta=sqrt((dn-1.d0)/(dn+1.d0));                    #row03
for (itn=0:maxitn)                                               #row04
   [f,g1]=calcfg(x); g=B'*g1; dg=norm(g);                        #row05
   if((mod(itn,intp)==0)&&(intp<=maxitn))                        #row06
     printf("itn %4d  f %14.6e\n",itn,f);                        #row07
   endif                                                         #row08
   if(r*dg<epsf) ist=1;  return; endif                           #row09
   xi=(1.d0/dg)*g; dx=B*xi;                                      #row10
   hs=r/(dn+1.d0); x-=hs*dx;                                     #row11
   B+=(beta-1.d0)*B*xi*xi';                                      #row12
   r=r/sqrt(1.d0-1.d0/dn)/sqrt(1.d0+1.d0/dn);                    #row13
endfor                                                           #row14
ist=4;                                                           #row15
endfunction                                                      #row16
```

TABLE 3. Results for minimizing f_1, f_2 by Algorithm **emshor**

	smooth function f_1			non-smooth function f_2		
n	k^*	$f(x^*_\varepsilon)$	$\|x^*_\varepsilon - x^*\|$	k^*	$f(x^*_\varepsilon)$	$\|x^*_\varepsilon - x^*\|$
10	3808	3.4e-19	3.9e-10	4484	8.2e-10	1.3e-10
20	15883	1.0e-18	4.3e-10	19044	4.7e-10	5.2e-11
50	104771	5.0e-19	3.1e-10	135113	6.9e-11	5.9e-13
100	454650	1.8e-19	6.9e-11	563705	5.3e-11	2.2e-12

operations with the symmetric matrix $H = BB^\top$. Results for applying this modified code to the minimization of function f_2 with $t = 2$, $\varepsilon = 1.0\text{e-}3$, $x_0 = 0$, and $r_0 \in \{5, 500\}$ are given in Table 4. In contrast to the other tables, `itn` denotes the total number of iterations performed, whereas `itr` (with `itr`≤`itn`) is the index, where the smallest value of f_2 was observed during the iteration process.

TABLE 4. Results for applying the H-form of Algorithm **emshor** to f_2

	$r_0 = 5$				$r_0 = 500$			
n	itn	$f(x_{\mathtt{itn}})$	itr	$f(x_{\mathtt{itr}})$	itn	$f(\mathtt{itn})$	itr	$f(x_{\mathtt{itr}})$
5	461	1.3e-03	446	1.0e-05	453	2.1e-01	443	2.0e-03
10	1664	3.1e-02	1467	8.3e-05	1767	2.4e+00	1690	2.0e-03
15	6541	6.5e-05	6528	2.9e-07	8615	5.6e-05	7804	3.1e-07
20	5627	5.7e+00	5356	2.4e-03	5434	1.1e+04	4980	2.1e-01

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- Стецюк П.І., Фішер А., Хомяк О.М. Уніфіковане представлення класичного методу еліпсоїдів. *Кібернетика та системний аналіз*. 2023. **59(5)**. С. 113-123.
- Fischer A., Khomyak O., Stetsyuk P. The ellipsoid method and computational aspects, *Commun. Optim. Theory* **21**. 2023. Pp. 1-14.
- Stetsyuk P.I., Fischer A., Khomiak O.M. “Ellipsoid methods with space scaling”. Proceedings of the 8th International Conference on Control and Optimization with Industrial Applications (COIA 2022). August 24-26, 2022, Baku, Azerbaijan. Pp. 402-404.

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