An exact nonsmooth penalty approach for a special class of linear programs

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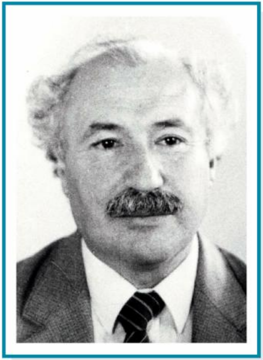
Content

We will use

r-algorithm for nonsmooth problems



Shor N.Z.

theorem for nonsmooth penalty functions

Pshenichny B.N.

Content

Special linear programming problem

We consider the problem

$$f_{max} = \max_{x \in \mathbb{R}^n} c^T x = \sum_{i=1}^{n} c_i x_i, \tag{1}$$

subject to

$$\sum_{i=1}^{n} a_{ij} x_i \leq b_j, \quad j = 1, \ldots, m, \tag{2}$$

$$0 \leq x_i \leq u_i, \quad i = 1, \ldots, n, \tag{3}$$

where $m >> n$, $\quad c_i \geq 0, \quad b_j > 0, \quad u_i < +\infty.$

The constraints in (3) allow us to bound f_{max} because

$$f_{max} \leq c^T u = \sum_{i=1}^{n} c_i u_i < +\infty. \tag{4}$$

Applications of linear programs ($m >> n$)

Examples are robust linear optimization problems, Danzig-Wolfe and Benders decomposition schemes, minimax and maximin problems, approximation problems with Chebyshev's minimax criterion, and Boolean estimation problems.

In robust linear optimization we consider

$$A(\xi)x \leq b(\xi) \quad \forall \xi \in U, \tag{5}$$

where the matrix $A(\xi)$, as well as the right-hand-side $b(\xi)$ may depend on $\xi \in U$. If the set U is finite, but contains up to thousands or millions of given parameters, we get linear programming problems with $m >> n$.

Independence (stability) number $\alpha(G)$

For an undirected graph G with vertices v_i, and associated variables x_i, the following integer program (IP) defines $\alpha(G)$:

$$\text{independence number} \quad \alpha(G) = \max \sum_{i=1}^{n} x_i, \tag{6}$$

$$\text{edge constraints} \quad x_i + x_j \le 1 \quad \text{for each edge} \quad v_i v_j, \tag{7}$$

$$\text{binary constraints} \quad x_i \in \{0, 1\} \quad \text{for each vertex} \quad v_i. \tag{8}$$

The independence number $\alpha(G)$ is the optimal value of IP (6)-(8) and, for an optimal solution $\{x_i\}$, the set of vertices $\{v_i : x_i = 1\}$ is a maximum independent set.

Clique independence number $\alpha_Q(G)$

The following special linear problem defines $\alpha_Q(G)$:

$$\alpha_Q(G) = \max \sum_{i=1}^{n} x_i, \tag{9}$$

subject to

$$\sum_{i \in V(Q)} x_i \le 1, \quad \text{for each maximal clique } Q(G), \tag{10}$$

$$0 \le x_i \le 1, \quad \text{for each vertex } v_i. \tag{11}$$

Tighter upper bounds for the number $\alpha(G)$

can be found by solving LP (9)-(11) with additional linear inequalities for odd cycles and p-wheels in the graph.

Content

A nonsmooth penalty function for LP

We rewrite problem (1)-(3) as the LP:

$$f_{max} = \max_{x \geq 0} c^T x : \quad Hx \leq \mathbf{1}, x \in \mathbb{R}^n, \tag{12}$$

where the $(m+n)\times n$ matrix $H = \{h_{ji}\}$ such that

$$h_{ji} = \begin{cases} a_{ji}/b_i, & j = 1, \ldots, m,\ i = 1, \ldots, n, \\ 1/u_i, & j = m+i,\ i = 1, \ldots, n, \\ 0, & j = m+i, \ldots, m+n,\ i = 1, \ldots, n, \end{cases}$$

$\mathbf{1}$ – $(m+n)$-vector, whose components are equal to one.

Theorem 1 (nonsmooth optimization problem)

Theorem 1

If $P > f_{max} > 0$, then the LP (12) is equivalent to the optimization problem

$$f_{max} = \max_{x \geq 0} f_P(x), \tag{13}$$

where $f_P(x)$ is defined by

$$f_P(x) = c^T x - P \cdot \max\left\{0, \max_{j=1,\ldots,m+n}\left\{\sum_{i=1}^{n} h_{ji}x_i - 1\right\}\right\}. \tag{14}$$

Proof of the Theorem 1

We used Pshenichnyi's theorem 2.14 ([3], p.25) for the LP

$$-f_{max} = \min_{x}\{-c^T x : Hx \leq \mathbf{1}, x \geq 0\}. \qquad (15)$$

The following penalty function is associated to (15):

$$\varphi_P(x) = -c^T x + P \cdot \max\left\{0, \max_{j=1,\dots,m+n}\left\{\sum_{i=1}^{n} h_{ji} x_i - 1\right\}\right\},$$

where $P > f_{max}$. Here $\varphi_P(x) = -f_P(x)$.

LPralg – r-algorithm for solving problem (13)

Nonsmooth optimization problem (Theorem 1):

$$f_{max} = \max_{x \geq 0} f_P(x), \quad \text{where} \quad P = \sum_{i=1}^{n} c_i u_i > f_{max} > 0. \quad (13)$$

Replacement of variables

The variables x_i are replaced by $|z_i|$, $i = 1, \ldots, n$. Then, instead of the function $f_P(x_1, \ldots, x_n)$, we obtain the function $f_P(|z_1|, \ldots, |z_n|)$, in which all local maxima are global.

Note: this does not require any selection of penalty parameters.

Content

Test problems

LP problem:

$$f_{max} = \sum_{i=1}^{n} c_i x_i, \tag{1}$$

subject to

$$\sum_{i=1}^{n} a_{ij} x_i \le b_j, \quad j = 1, \dots, m, \tag{2}$$

$$0 \le x_i \le 1, \quad i = 1, \dots, n, \tag{3'}$$

where

$$c_i \sim U(0,1), \;\; a_{ji} \sim U(0,1), \;\; b_j = 0.9 \sum_{i=1}^{n} a_{ji}.$$

Parameters for nonsmooth problems

$$f_{max} = \max_{x \geq 0} f_P(x), \quad P = \sum_{i=1}^{n} c_i.$$

Parameters of the r-algorithm:

$x := 0 \in \mathbb{R}^n$, $alpha := 4$, $q1 := 1.0(0.99)$, $h0 := 1.0$.

Processor and programming language:

Intel Core i5-9400f, 2.9 GHz, 16GB RAM, GNU Octave 5.1.0

Results for GLPK and LPralg (runtimes in seconds)

n	m	t_{GLPK}	t_{LPralg}	t_{GLPK}/t_{LPralg}	Δ
	100.000	1.80	0.73	2.47	2.9e-08
25	500.000	8.55	4.90	1.75	9.2e-09
	1.000.000	18.0	12.2	1.47	2.0e-08
	100.000	4.00	2.44	1.64	4.6e-08
50	500.000	22.4	15.2	1.48	4.3e-08
	1.000.000	46.2	26.3	1.75	2.2e-08
	100.000	12.8	13.2	0.97	1.2e-07
100	500.000	66.7	45.5	1.46	1.3e-07
	1.000.000	284.5	110.0	2.59	2.4e-07

The GLPK solver needs in average 0.97–2.59 times longer for the test problems than LPralg, $\Delta = \|x_{GLPK} - x_{LPralg}\|$.

Runtimes for LPralg (in seconds)

n	m	$q_1 = 0.99$	$q_1 = 1.00$
100	3.000.000	391.3	338.7
50	6.000.000	286.1	274.0
25	12.000.000	157.9	110.9

Memory required: 2.4 GB

$$n \times m = 300.000.000$$

$$300.000.000 \times 8 = 2.4 \text{ GB}$$

Conclusion

A new solution method

is based on the reformulation of the linear program as an equivalent non-smooth optimization problem subject to non-negativity constraints.

For large randomly generated test problems

with much more constraints than variables the competitiveness with the GLPK solver is demonstrated.

References

1. Goryashko A., Nemirovski A. **Robust energy cost optimization of water distribution system with uncertain demand**. *Automation and Remote Control*. 2014, 75(10), pp. 1754–1769.

2. Stetsyuk P.I., Lykhovyd A.P. **LP-oriented upper bounds for the weighted stability number of a graph**. *Cybernetics and Systems Analysis*. 2009, 45, pp. 141–152.

3. Pshenichnyi B.N. **The Linearization Method**. Moscow, Nauka, 1983. (in Russian).

References

4. Shor N.Z. **Nondifferentiable Optimization and Polynomial Problems**. Amsterdam, Kluwer, 1998.

5. Stetsyuk P., Fischer A., Pichugina O. **A penalty approach to linear programs with many two-sided constraints**. *In: P. Pardalos, M. Khachay, A. Kazakov (Eds.) Mathematical Optimization Theory and Operations Research. MOTOR 2021. Lecture Notes in Computer Science, Springer, Cham.* 2021, vol 12755, pp. 206-217.

6. Free Software Foundation Inc.: GNU Linear Programming Kit (GLPK). http://www.gnu.org/software/glpk/glpk.html

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Questions?

THANK YOU FOR YOUR ATTENTION

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