1d and 2d Ellipsoids in Shor's Algorithms

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Outline

1. On the ellipsoid method
2. r-algorithms and Shor's Dream
3. 2d-ellipsoid and r-algorithms
4. On using 2d-ellipsoid

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Ellipsoid method was proposed

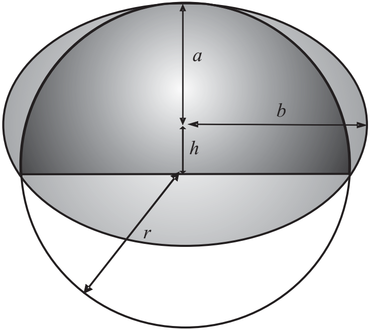
1976 by **Yudin and Nemirovskii** as a method of successive cutting-plane [1].

1977 by **Shor** as a variant of the method with space dilation in the direction of the subgradient [2].

1. Yudin D.B. and Nemirovskii A.S. *Informational complexity and effective methods for the solution of convex extremal problems* // Ekonom. Mat. Metody, 12, No. 2 (1976).

2. Shor N.Z. *Cut-off method with space extension in convex programming problems* // Cybernetics, 13, No. 1 (1977).

The idea of ellipsoid method (1d-ellipsoid $\mathcal{E}_n$)



The 1d-ellipsoid $\mathcal{E}_n$, containing half of ball S_n in E^n, has minimal volume if

$$a = \frac{n}{n+1} r, \;\; b = \frac{n}{\sqrt{n^2-1}} r, \;\; h = \frac{1}{n+1} r.$$

To transform $\mathcal{E}_n$ into a ball we have to dilate the space with coefficient $\alpha = \frac{b}{a} = \sqrt{\frac{n+1}{n-1}}$.

The ratio of $\mathcal{E}_n$ volume to S_n volume equals

$$q(n) = \frac{\mathrm{vol}(\mathcal{E}_n)}{\mathrm{vol}(S_n)} = \frac{a}{r} \left(\frac{b}{r}\right)^{n-1} = \sqrt{\frac{n-1}{n+1}} \left(\frac{n}{\sqrt{n^2-1}}\right)^n \leq 1 - \frac{1}{2n}$$

Operator of space dilation

is introduced by N.Z. Shor (1969) and has the following form

$$R_\alpha(\xi) = I_n + (\alpha - 1)\xi\xi^T, \quad \text{where} \quad \alpha > 1.$$

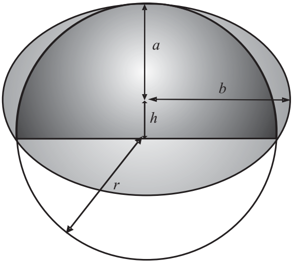
Here: α is the coefficient of space dilation in the normed direction $\xi \in E^n$, $\|\xi\| = 1$; I_n is the identity $n \times n$-matrix.

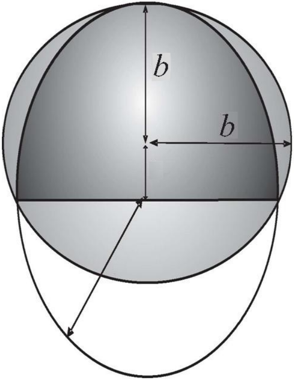
Shor's algorithms use the „inverse“ operator

$$R_\beta(\xi) = I_n + (\beta - 1)\xi\xi^T, \quad \text{where} \quad \beta = \frac{1}{\alpha} < 1,$$

which means "compression" of subgradients space.

1d-ellipsoid before and after space dilation





The 1d-ellipsoid $\mathcal{E}_n$ is a ball in the dilated space.
The radius of this ball equals „b“.

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Мечта Шора актуальна и сегодня ...

„Теория всего класса алгоритмов с растяжением пространства далека от совершенства. Нам кажется достаточно реалистичной целью – построение такого алгоритма, который по своей практической эффективности не уступал бы r-алгоритму и был столь же хорошо обоснован как метод эллипсоидов" [*].

* Гершович В.И., Шор Н.З. *Метод эллипсоидов, его обобщения и приложения.* // Кибернетика, 1982, № 5.

Shor's Dream is still actual today ...

„The theory of the whole class of algorithms with space dilation is far from perfect. It seems quite realistic goal – building of such an algorithm, which, by its practical efficiency, was not inferior to r-algorithm and would be just as well founded as the ellipsoid method" [*].

* **V.I. Gershovich, N.Z. Shor** *Method of ellipsoids, its generalizations and applications* // Cybernetics, 18, No 5 (1982).

r-Algorithms and ellipsoids (Stetsyuk,1996)

An attempt to explain r-algorithms was made in the paper

P.I. Stetsyuk. "r-Algorithms and ellipsoids," Cybernetics and System Analysis, **32**, No. 1, 93–110 (1996).

Here, the transformation of a special ellipsoid into a ball uses anti-zigzag method like one in r-algorithms.

Space dilation is realized in the direction of the difference of two normed successive subgradients. This direction is close to the direction of the difference of two successive subgradients if the norms of the subgradients are approximately equal.

Lemma (Stetsyuk, 1996)

Let B_k *be* $n \times n$*-matrix, such that* $\|B_k^{-1}(x_k - x^*)\| \le r$*; let* g_1 *and* g_2 *be* n*-dimensional vectors, such that* $(x_k - x^*, g_1) \ge 0$, $(x_k - x^*, g_2) \ge 0$ *and* $(B_k^T g_1, B_k^T g_2) < 0$*. If matrix* B_{k+1} *is recalculated according to the rule*

$$B_{k+1} = B_k R_{\beta_1}\left(\frac{\xi - \eta}{\|\xi - \eta\|}\right) R_{\beta_2}\left(\frac{\xi + \eta}{\|\xi + \eta\|}\right), \ \xi = \frac{B_k^T g_1}{\|B_k^T g_1\|}, \eta = \frac{B_k^T g_2}{\|B_k^T g_2\|},$$

where $\beta_1 = \sqrt{1 + (\xi, \eta)}$ *and* $\beta_2 = \sqrt{1 - (\xi, \eta)}$*, then matrix* B_{k+1} *has the following properties: (i)* $\|B_{k+1}^{-1}(x_k - x^*)\| \le r$*;*
(ii) $\det(B_{k+1}) = \det B_k \sqrt{1 - (\xi, \eta)^2}$*; (iii)* $(B_{k+1}^T g_1, B_{k+1}^T g_2) = 0$*.*

The lemma interpretation

We have:

(1) point x^* localized within ellipsoid $\|B_k^{-1}(x_k - x^*)\| \le r$;
(2) g_1 and g_2 – normals to cutting-planes through x_k;
(3) obtuse angle φ_k between normals in $Y_k = A_k X = B_k^{-1} X$.

After dilation by „normals difference"

we receive:

(i) localization of x^* within ellipsoid $\|B_{k+1}^{-1}(x_k - x^*)\| \le r$;
(ii) ellipsoid volume decrease by $1/\sqrt{1 - \cos^2 \varphi_k}$ times;
(iii) g_1, g_2 transformed into orthogonal in $Y_{k+1} = A_{k+1} X$.

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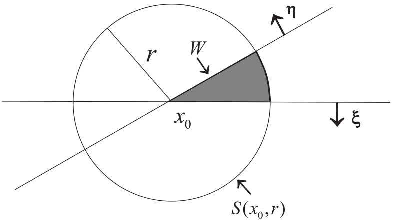
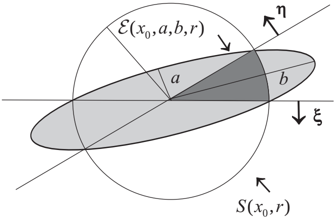
The convex set W and 2d-ellipsoid $\mathcal{E}(x_0, a, b, r)$





the set W is the intersection of a ball $S(x_0, r)$ and half-spaces $P(x_0, \xi)$, $P(x_0, \eta)$

2d-ellipsoid contains the set W and has the minimum volume

Stetsyuk (1996) r-Algorithms and ellipsoids, Cybernetics and System Analysis, **32**, No. 1.

Properties of 2d-ellipsoid $\mathcal{E}(x_0, a, b, r)$

2d-ellipsoid has the following parameters:

$$a = r\sqrt{1 + (\xi, \eta)} < r; \quad b = r\sqrt{1 - (\xi, \eta)} > r.$$

(i) If $(\xi, \eta) < 0$ then 2d-ellipsoid contains the convex set W.

(ii) The ratio of 2d-ellipsoid volume to ball volume equals

$$q = \frac{\text{vol}(\mathcal{E}(x_0, a, b, r))}{\text{vol}(S(x_0, r))} = \left(\frac{a}{r}\right)\left(\frac{b}{r}\right) = \sqrt{1 - (\xi, \eta)^2} < 1.$$

If the angle between vectors ξ and η becomes closer to π (180 degrees) then the ratio q becomes smaller.

Transformation of 2d-ellipsoid into a ball

corresponds to the updating of matrix

$$B_{k+1}^{-1} = R_{\alpha_2}\left(\frac{\xi+\eta}{\|\xi+\eta\|}\right) R_{\alpha_1}\left(\frac{\xi-\eta}{\|\xi-\eta\|}\right) B_k^{-1},$$

i.e. we dilate space in the direction of $\frac{\xi-\eta}{\|\xi-\eta\|}$ with

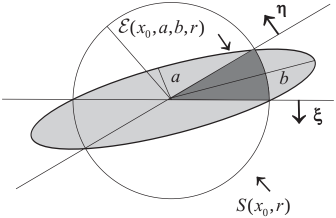
$$\alpha_1 = \frac{r}{a} = \frac{1}{\sqrt{1+(\xi,\eta)}} > 1,$$

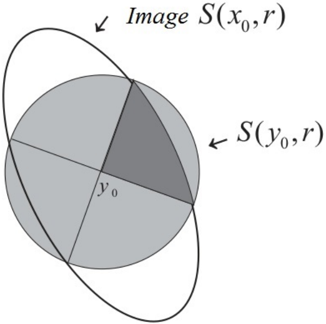
then we „dilate“ space in the direction of $\frac{\xi+\eta}{\|\xi+\eta\|}$ with

$$\alpha_2 = \frac{r}{b} = \frac{1}{\sqrt{1-(\xi,\eta)}} < 1.$$

2d-ellipsoid before and after transformation

In the transformed space $Y = R_{\alpha_2}(\frac{\xi+\eta}{\|\xi+\eta\|}) R_{\alpha_1}(\frac{\xi-\eta}{\|\xi-\eta\|}) X \equiv E^n$





2d-ellipsoid $\mathcal{E}(x_0, a, b, r)$

becomes a ball $S(y_0, r)$

Closeness to Shor's r-algorithm

(iii) Vectors ξ and η become orthogonal in the transformed space.

This feature allows to „extend" cone of feasible directions of the function decrease for the subgradient process in the transformed space of variables, as in Shor's r-algorithm.

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What can we use 2d-ellipsoid for?

for construction of accelerated ellipsoid methods

for the following problems:

1) convex programming problems;

2) finding saddle points of concave-convex functions;

3) special cases of variational inequalities, linear and non-linear complimentarity problems.

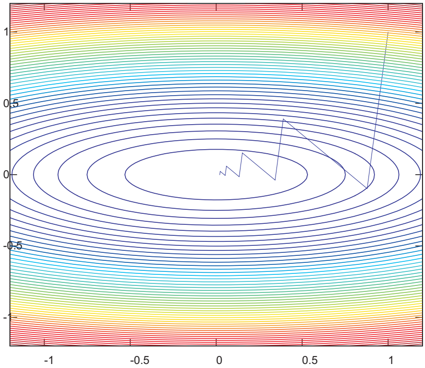
What can we expect from such methods?

The rate of convergence is close to the rate of r-algorithms.

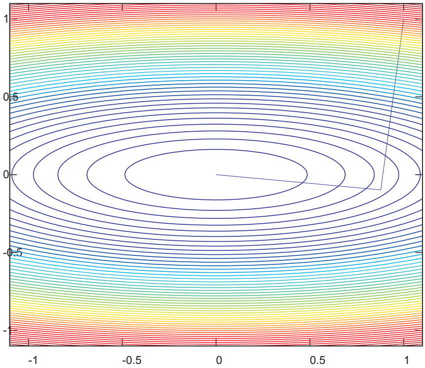
P.I. Stetsyuk (2014) *Ellipsoid methods and r-algorithms (in Russian)*. Chisinau: Evrika, 488 p.

It is proven by subgradient methods with space transformation, which speed up the well-known Polyak's subgradient method. They are effective for ill-conditioned functions.

Example 1 (quadratic function)

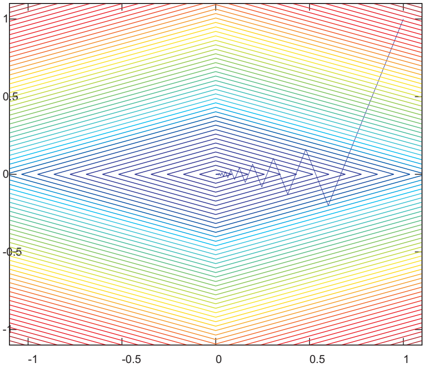


1.1 Trajectory of Polyak's method

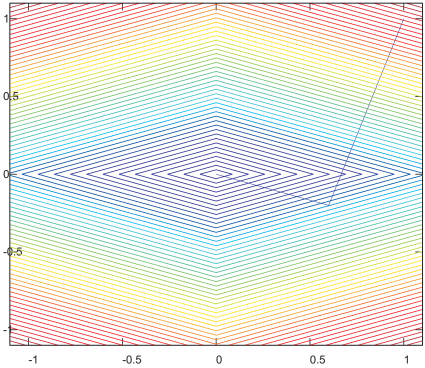


1.2...accelerated Polyak's method

Example 2 (piecewise linear function)

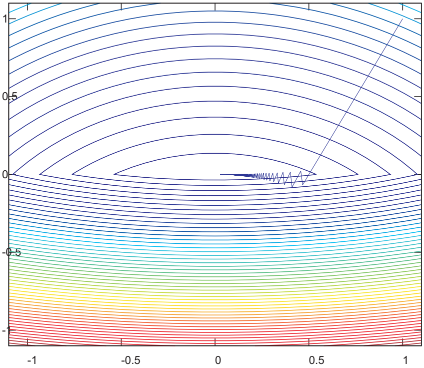


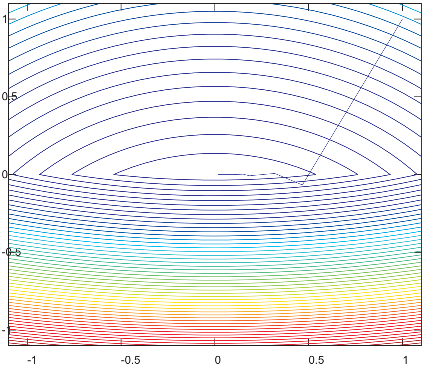
2.1 Trajectory of Polyak's method



2.2...accelerated Polyak's method

Example 3 (piecewise quadratic function)





3.1 Trajectory of Polyak's method

3.2 ...accelerated Polyak's method

Conclusion

The accelerated variants of ellipsoid methods on

the basis of the 1d-ellipsoid and the 2d-ellipsoid can be used for solving a variety of problems: convex programming problems, problems of finding saddle points of convex-concave functions, and special cases of variational inequalities.

Thanks

Volkswagen Foundation
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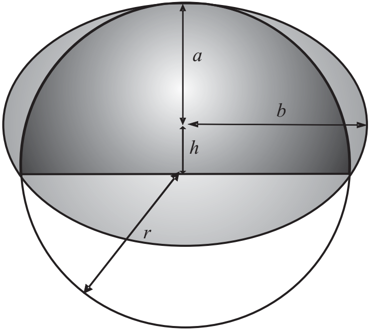
Questions?

THANK YOU
FOR YOUR ATTENTION!

BACKUP SLIDES: 1d-ellipsoid (Shor, 1977)



Minimal volume 1d-ellipsoid $\mathcal{E}_n$, containing half-ball in E^n, has parameters

$$a = \frac{n}{n+1} r, \;\; b = \frac{n}{\sqrt{n^2-1}} r, \;\; h = \frac{1}{n+1} r.$$

Dilating the space by a factor of

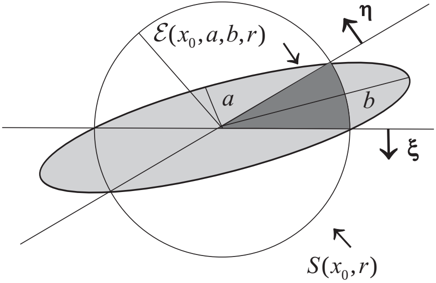
$$\alpha = \frac{b}{a} = \sqrt{\frac{n+1}{n-1}}$$

we transform $\mathcal{E}_n$ into ball.

The ratio of $\mathcal{E}_n$ volume to S_n volume equals

$$q(n) = \frac{\text{vol}(\mathcal{E}_n)}{\text{vol}(S_n)} = \frac{a}{r}\left(\frac{b}{r}\right)^{n-1} = \sqrt{\frac{n-1}{n+1}}\left(\frac{n}{\sqrt{n^2-1}}\right)^n \le 1 - \frac{1}{2n},$$

BACKUP SLIDES: 2d-ellipsoid (Stetsyuk, 1996)



The space transformation is two space dilations:

in the direction $\frac{\xi-\eta}{\|\xi-\eta\|}$ with

$$\alpha_1 = \frac{r}{a} = \frac{1}{\sqrt{1+(\xi,\eta)}} > 1,$$

in the direction $\frac{\xi+\eta}{\|\xi+\eta\|}$ with

$$\alpha_2 = \frac{r}{b} = \frac{1}{\sqrt{1-(\xi,\eta)}} < 1.$$

$$q = \frac{\mathrm{vol}(\mathcal{E}(x_0,a,b,r))}{\mathrm{vol}(S(x_0,r))} = \left(\frac{a}{r}\right)\left(\frac{b}{r}\right) = \sqrt{1-(\xi,\eta)^2}.$$