Subgradient method with Polyak's step in transformed space

Petro Stetsyuk[1], Viktor Stovba[1]

[1] V.M. Glushkov Institute of Cybernetics, Kyiv, Ukraine

Norwegian-Eurasian Workshop on New Resilience Challenges in Ecological-Economic Problems at the Digital Era

Kyiv, Ukraine
September 22 – 27, 2019

Outline

1. Polyak's subgradient method ($m = 1$)
2. Method A ($m \geqslant 1$)
3. Method B ($m \geqslant 1$, matrix B)
4. Comparison of the methods A and B

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Formulation of the problem

We consider the following problem

$$\text{to find} \quad x^\star = \arg\min_{x \in R^n} f(x) \quad \text{if } f^\star \text{ is known,} \qquad (1)$$

where $f(x)$ is a convex function and $f^\star = f(x^\star) = \min_{x \in R^n} f(x)$.

Main inequality for the subgradient

For convex function $f(x)$ the following inequality is valid

$$(x - x^\star, g_f(x)) \geq f(x) - f^\star, \quad \forall x \in R^n, \tag{2}$$

where $g_f(x)$ is a subgradient (gradient) of the function $f(x)$.

Inequality (2) is used for the calculation of the step in subgradient method, which B.T. Polyak offered in 1969.

Polyak B.T. Minimization of unsmooth functionals // *USSR Comput. Math. Math. Phys.*. 1969. Vol. 9, No. 3, pp. 14-29.

Polyak's subgradient method

Polyak's subgradient method has the iterative form

$$x_{k+1} = x_k - h_k \frac{g_f(x_k)}{\|g_f(x_k)\|}, \quad h_k = \frac{f(x_k) - f^\star}{\|g_f(x_k)\|}, \quad k=0, 1, \ldots \tag{3}$$

Step h_k is called Polyak's step.

Decrease of the distance to the minimum point

Theorem 1 (Polyak, 1969)

The sequence $\{x_k\}_{k=0}^{\infty}$, *generated by the method (3), satisfies the inequalities*

$$\|x_{k+1} - x^\star\|^2 \leq \|x_k - x^\star\|^2 - \frac{(f(x_k) - f^\star)^2}{\|g_f(x_k)\|^2}, \quad k = 0, 1, 2, \ldots$$

Remark. Theorem 1 guarantees that in Polyak's method the distance to the minimum point decreases monotonically.

m-inequality for the subgradient

We consider methods A and B for convex functions $f(x)$, subgradients $g_f(x)$ of which satisfy the following condition:

$$(x - x^\star, g_f(x)) \geq m(f(x) - f^\star), \quad \forall x \in R^n, \tag{4}$$

where parameter $m \geq 1$.

The method A uses the step h_k in **original space**, the method B uses h_k in **transformed space** of variables.

Contents

Method A

Initialization. $f^\star$, $m \geq 1$, $x_0 \in R^n$, $\varepsilon > 0$.

A1. Calculate $f(x_k)$ and $g_f(x_k)$.
If $f(x_k) - f^* \leq \varepsilon$, then STOP ($k^* = k$, $x^*_\varepsilon = x_k$).

A2. Calculate the next point

$$x_{k+1} = x_k - h_k \frac{g_f(x_k)}{\|g_f(x_k)\|}, \quad h_k = \frac{m(f(x_k) - f^*)}{\|g_f(x_k)\|}.$$

A3. Go to the $(k+1)$-th iteration with x_{k+1}.

Decrease of the distance to the minimum point

Theorem 2.1 (Stetsyuk, 2012)

The sequence $\{x_k\}_{k=0}^{k^*-1}$, *generated by the method A, satisfies the inequalities*

$$\|x_{k+1} - x^\star\|^2 \leq \|x_k - x^\star\|^2 - \frac{m^2(f(x_k) - f^\star)^2}{\|g_f(x_k)\|^2}, \quad k = 0, 1, 2, \ldots$$

Remark. Theorem 2.1 guarantees that in Polyak's subgradient method the distance to the minimum point decreases monotonically if the inequality (4) is used.

Convergence rate of the method A

Theorem 2.2 (general case)

If a convex function $f(x)$ satisfies the inequality (4) the following inequality is true:

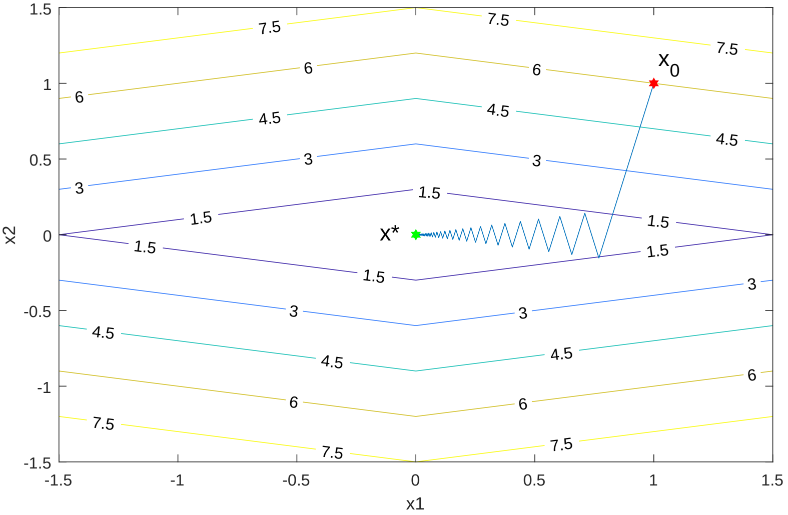
$$\lim_{k \to \infty} \sqrt{k}(f(x_k) - f^*) = 0 \tag{5}$$

Theorem 2.3 (acute minimum case)

Let function $f(x)$ satisfy the inequality $f(x) - f^* \geq \alpha \|x - x^*\|$. Then the method A converges with geometric progression rate with common ratio $q_1 = 1 - (\frac{m\alpha}{c_1})^2$, where c_1 is the constant that bounds subgradient $g_f(x)$ norm.

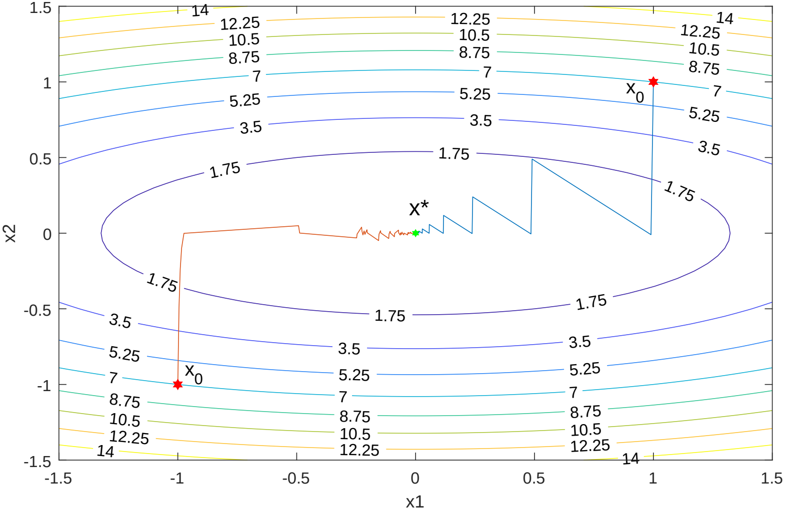
Method A for $f_1(x_1, x_2) = |x_1| + 5|x_2|$

$$m = 1,\ \ f^\star = 0,\ \ x_0 = (1, 1)^T$$



Method A for $f_2(x_1, x_2) = x_1^2 + 6x_2^2$

Blue - $m = 2,\ x_0 = (1, 1)^T$ Red - $m = 1,\ x_0 = (-1, -1)^T$



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Method B

Initialization. $f^\star$, $m \geq 1$, $x_0 \in R^n$, $n \times n$ matrix B, $\varepsilon > 0$.

A1. Calculate $f(x_k)$ and $g_f(x_k)$.
If $f(x_k) - f^* \leq \varepsilon$, then STOP ($k^* = k$, $x^*_\varepsilon = x_k$).

A2. Calculate the next point

$$x_{k+1} = x_k - h_k B \frac{B^T g_f(x_k)}{\|B^T g_f(x_k)\|}, \quad h_k = \frac{m(f(x_k) - f^*)}{\|B^T g_f(x_k)\|}.$$

A3. Go to the $(k+1)$-th iteration with x_{k+1}.

Here h_k is the Polyak's step in the **transformed** space of variables.

If B is an identity matrix, the method B turns into the method A.

Decrease of the distance to the minimum point

Theorem 3.1 (Stetsyuk, Stovba, Chernousova, 2018)

The sequence $\{x_k\}_{k=0}^{k^*-1}$*, generated by the method B, satisfies the inequalities*

$$\|A(x_{k+1} - x^\star)\|^2 \leq \|A(x_k - x^\star)\|^2 - \frac{m^2(f(x_k) - f^\star)^2}{\|B^T g_f(x_k)\|^2}, \quad k = 0, 1, \ldots$$

Remark. Theorem 3.1 guarantees that in the method B the distance to the minimum point decreases monotonically in the trasformed space if the inequality (4) is used.

Convergence rate of the method B

Theorem 3.2 (general case)

If a convex function $\varphi(y) = f(By) = f(x)$ *satisfies the inequality* $(y - y^*, g_\varphi(y)) \geq m(\varphi(x) - \varphi^*)$ *for any* y*, the following inequality is true:*

$$\lim_{k\to\infty} \sqrt{k}(\varphi(y_k) - \varphi^*) = 0 \tag{6}$$

Theorem 3.3 (acute minimum case)

Let function $\varphi(y)$ *satisfy the inequality* $\varphi(y) - \varphi^* \geq \alpha\|y - y^*\|$*. Then the method B converges with geometric progression rate with common ratio* $q_2 = 1 - (\frac{m\alpha}{c_2})^2$*, where* c_2 *is the constant that bounds subgradient* $g_\varphi(y)$ *norm.*

Method B for $f_1(x_1, x_2) = |x_1| + 10|x_2|$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1/\alpha \end{pmatrix}, \quad x_0 = (1, 1)^T$$

ε_f	$\alpha = 1$	$\alpha = 1.5$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$	$\alpha = 5$
1.0e-02	262	114	62	19	17	13
1.0e-04	492	216	119	44	31	22
1.0e-06	**722**	**319**	**177**	**70**	**45**	**31**
1.0e-08	952	421	234	95	60	40
1.0e-10	1183	523	292	121	74	49

Contents

Functions used to compare the methods A and B

1. Piecewise quadratic function

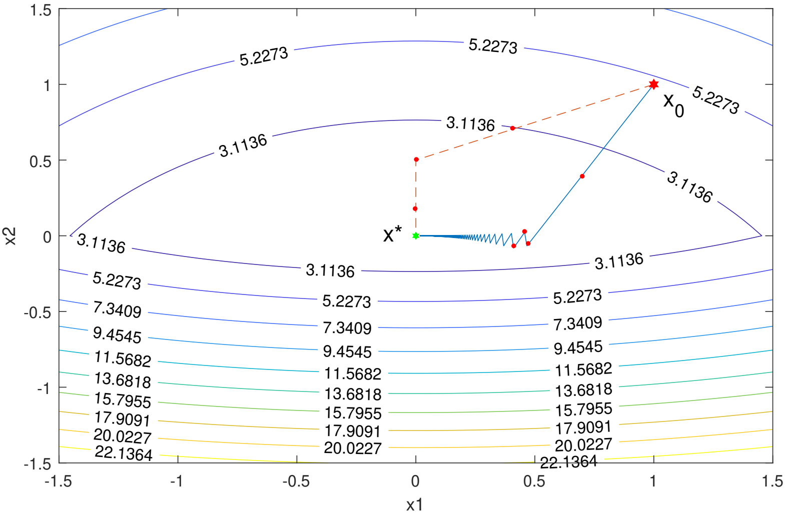
$$f_3(x_1, x_2) = \max\left\{x_1^2 + (2x_2 - 2)^2 - 3, x_1^2 + (x_2 + 1)^2\right\},$$
$$x^* = (0, 0),\ \ f^* = 1.$$

2. Quadratic function

$$f_4(x) = \|Ax - b\|^2,\ \ x^* = (1, 1, ..., 1)^T,\ \ f^* = 0.$$

Methods A and B for piecewise quadratic function

$x_0 = (1, 1)^T$, $B = diag(1; 0.5)$, method A - blue, method B - red



Methods A and B for function $f_4(x) = \|Ax - b\|^2$

$$A = \|a_{ij}\|_{i,j=1}^{l,n} = \left\| \begin{array}{ccc} 100 & 0 & 0...0 \\ 0 & 100 & 0...0 \\ & A_1 & \end{array} \right\|, \quad b_i = \sum_{j=1}^{n} a_{ij},\, i = \overline{1, l},$$

A_1 - 498×100 random matrix, $m = 2$.

ε	Method A		Method B	
	itnA	$\|x^*_\varepsilon - x^*\|$	**itnB**	$\|x^*_\varepsilon - x^*\|$
1.0e-04	695	8.0612e-04	86	1.2175e-04
1.0e-06	1085	8.2861e-05	108	1.1773e-05
1.0e-08	1491	8.3470e-06	130	1.1353e-06
1.0e-10	**1901**	**8.3881e-07**	**150**	**1.3526e-07**
1.0e-12	2313	8.3994e-08	172	1.3018e-08
1.0e-14	2725	8.4440e-09	194	1.2523e-09

Conclusions

1. Using parameter $m \geq 1$ in Polyak's subgradient method gives a possibility to minimize special classes of convex functions more effectively.

2. Application of space transformation can accelerate convergence of subgradient method with Polyak's step in the transformed space of variables.

References

- *Agmon S.* The relaxation method for linear inequalities // Canad. J. Math. 6 (1954), 382–392.
- *Motzkin T., Schoenberg I.J.* The relaxation method for linear inequalities // Canad. J. Math. 6 (1954), 393–404.
- *Еремин И.И.* Обобщение релаксационного метода Агмона-Моцкина. УМН, 1965. Вып. 2 (122). С. 183–187.
- *Поляк Б.Т.* Минимизация негладких функционалов. Ж. вычисл. матем. и матем. физ., 1969. Том №9 (3). С. 509–521.

References

- *Поляк Б.Т.* Введение в оптимизацию. Наука, 1983. С. 384. (перевидано у 2014 р.)
- *Стецюк П.И.* Ускорение субградиентного метода Поляка. Теорія оптимальних рішень, 2012. №11. С. 151-160.
- *Stetsyuk P., Stovba V., Chernousova Z.* Subgradient method with Polyak's step in transformed space. Optimization and Applications. OPTIMA 2018. Vol. 974. Springer, Cham. P. 49-63.
- *Стецюк П.І., Стовба В.О., Жмуд О.О.* Про швидкість збіжності субградієнтних методів з кроком Поляка. Наук. вісник Ужгород. ун-ту, 2019. Вип. №1 (34). С. 94-101.

Thanks

Volkswagen Foundation
grant No 90 306

Questions?

THANK YOU
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