



Using Arbitrary Norms in Packing Problems

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ПРИВІТАННЯ



Вельмишановний професоре Паносе М. Пардалос!

У день Вашого 70-літнього ювілею прийміть щирі й теплі вітання від наукового колективу Інституту кібернетики імені В.М. Глушкова Національної академії наук України.

Ви є всесвітньо відомим вченим в галузі глобальної та комбінаторної оптимізації, дослідження операцій. Ваші численні досягнення визнаються та цінуються світовою академічною спільнотою.

Ви зробили безцінний внесок у розвиток української науки як координатор багатьох спільних міжнародних проєктів, які виконувались вченими Національної академії наук України.

Бажаємо Вам міцного здоров'я, невичерпної енергії та творчого наукового натхнення для досягнення нових фантастичних висот у всіх сферах Вашого життя.

З глибокою повагою,

директор
Інституту кібернетики
імені В.М. Глушкова
НАН України



Іван Сергієнко



GREETINGS



Dear Professor Panos M. Pardalos!

On the day of your 70th anniversary, please, accept sincere and warm congratulations from the research team of V.M. Glushkov Institute of Cybernetics of the National Academy of Sciences of Ukraine.

You are a world-renowned scientist in the field of global and combinatorial optimization, operations research. Your achievements are well respected and valorized by the world academic community.

You made a priceless contribution to the development of Ukrainian science as the coordinator of many joint international projects, which were carried out by researchers from the National Academy of Sciences of Ukraine.

We wish you strong health, unlimited energy, and creative scientific inspiration for new fantastic achievements in all spheres of your life.

With deep respect,

Director of V.M. Glushkov
Institute of Cybernetics
of the NAS of Ukraine



Ivan Sergienko



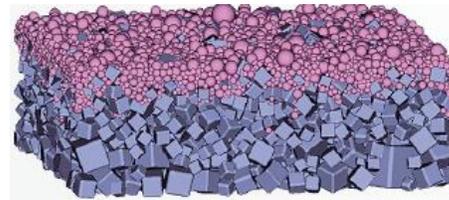
We discuss here:

- Packing problems and their applications
- Packing generalized spheres defined by arbitrary norms (PCGS)
- Three classes of PCGS Problem
 - Generalized Balance Packing Problem (GBPP)
 - Generalized Homothetic Packing Problem (GHPP)
 - Generalized Sparse Packing Problem (GSPP)
- Computational results for spheres defined by L_p norms
- Conclusions and future research

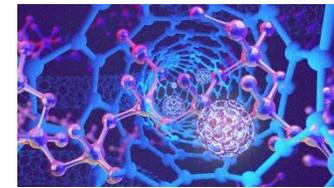
Applications: Packing problems have a wide spectrum of applications including: materials science, additive manufacturing, medicine, biology, space engineering,...



3D printing



Geology



Nanotechnology



Materials science



Chemistry production



Space engineering



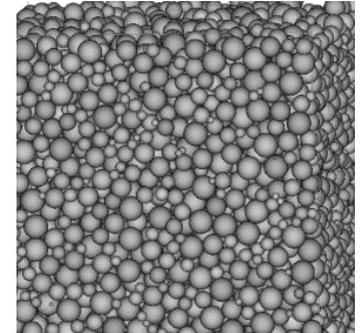
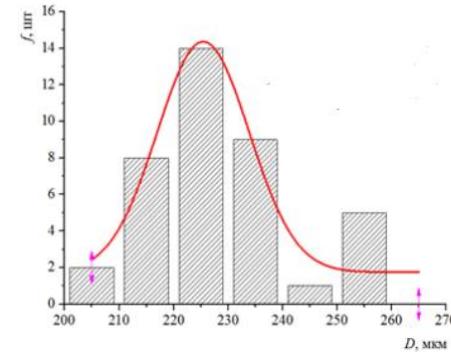
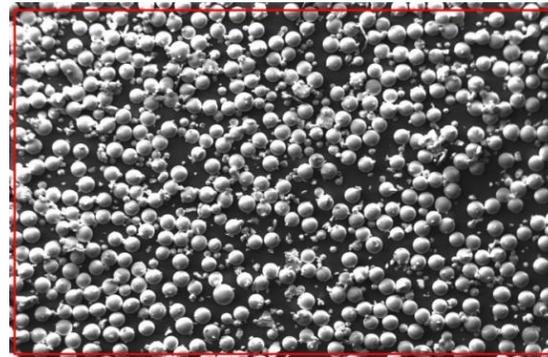
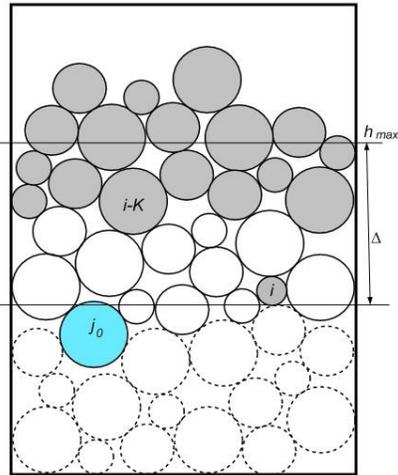
Applications: Optimization of microstructure design of powder alloys for additive technologies

To obtain high-quality and durable parts by 3D printing, specific characteristics (porosity and proportion of various sizes particles) in the mixture used for printing or sintering must be assured.

To predict these characteristics, we develop mathematical models and solution algorithms for finding optimized packing spherical/ non-spherical particles in a 3D volume.

Applications: Optimal filling of a given volume by spherical titanium alloy powder particles

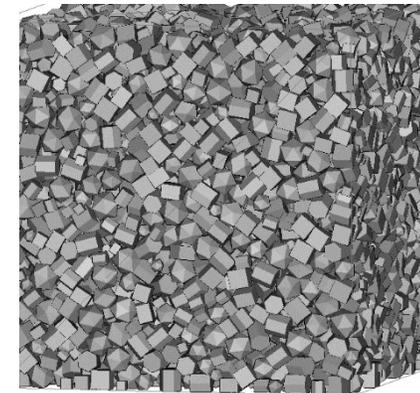
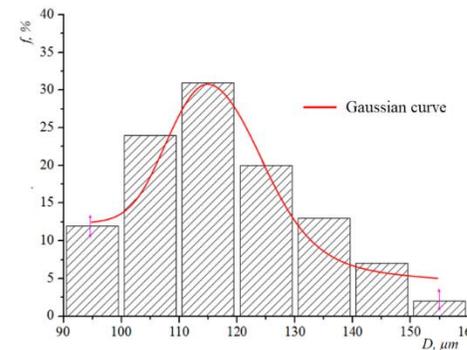
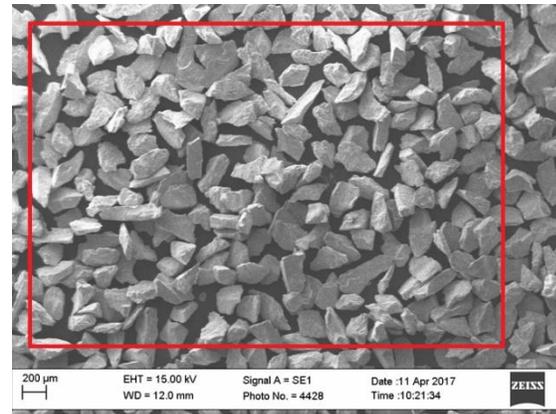
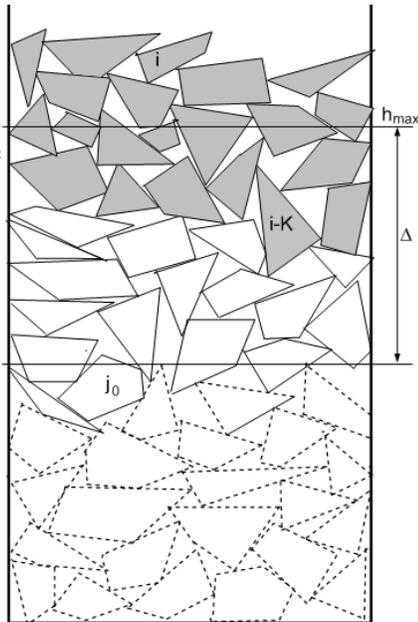
Packing spherical particles of the fraction from 200 to 250 μm . Porosity 34.177%



The difference between experimental and computational data is 1%-2%

Applications: Optimal filling of a given volume by non-spherical titanium alloy powder particles

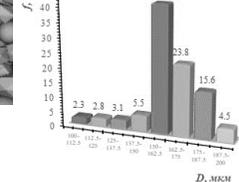
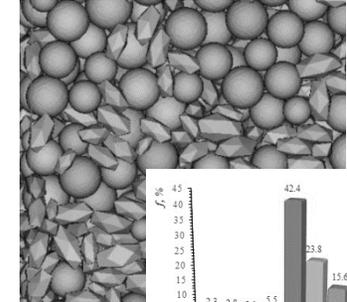
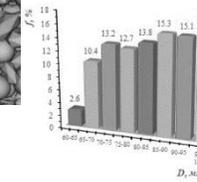
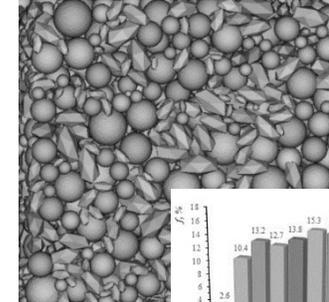
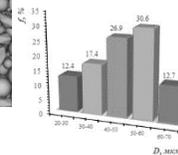
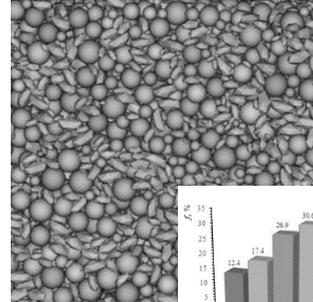
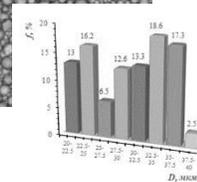
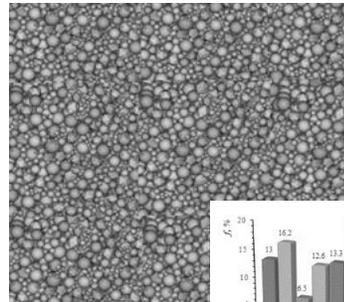
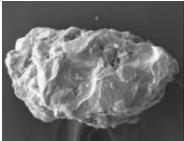
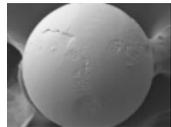
Packing non-spherical particles of the fraction from 100 to 160 μm . Porosity 31.33%



The difference between experimental and computational data is 5%-7%

Applications: Optimal filling of a given volume by mixtures of spherical and non-spherical titanium alloy powder particles

Packing mixtures of the fraction from 200 to 400 μm . Porosity 32.3%



The difference between experimental and computational data is 2%-7%

Applications: 3D printing - Optimizing the arrangement of the components on a building platform for 3D printing



The number of components manufactured in one or many printing jobs **can be considerably increased.**

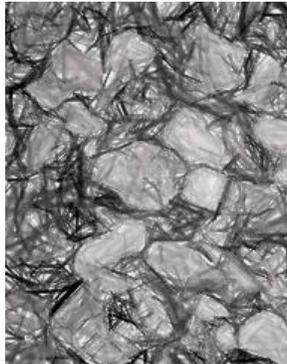
This results directly in a significant decrease in the manufacturing costs for the components.

For the selected example of geometries and boundary conditions our approach allows:

- **an increase in building platform utilization of 70%** and
- **a decrease in the number of printing jobs by 44%** have been achieved.

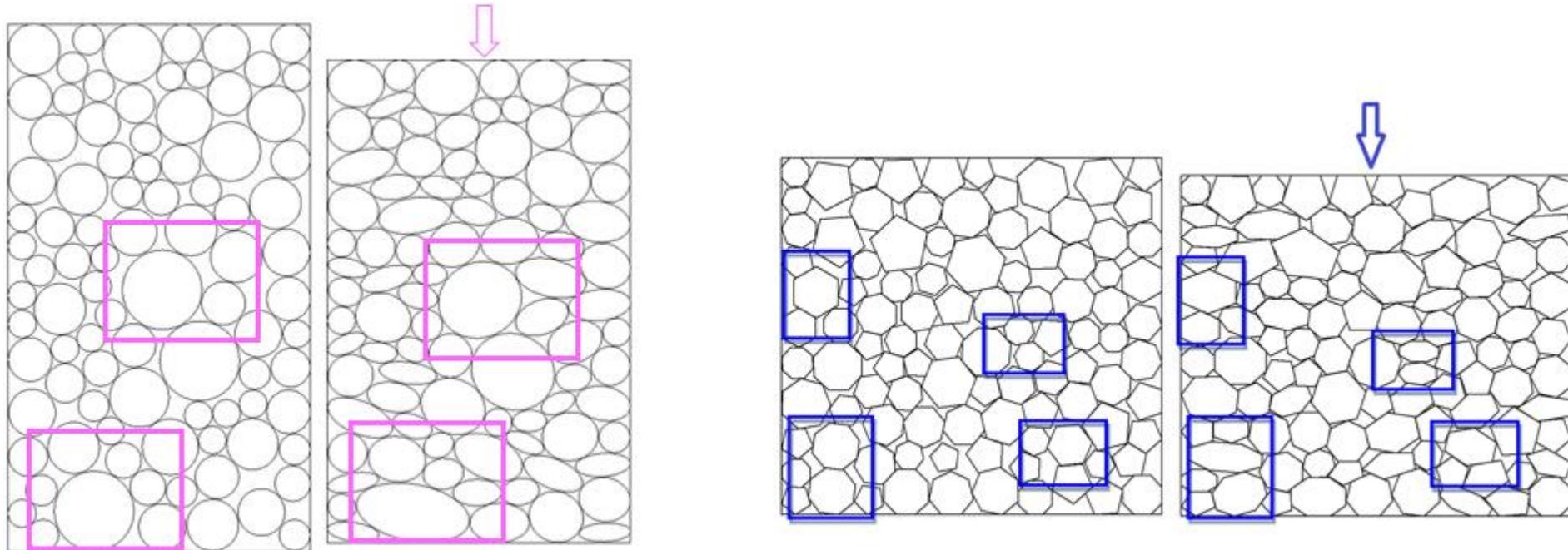
Applications: Modeling porous media under pressure

Soft packing problems are motivated, by modeling porous media under pressure and arise, e.g., in the oil and gas extracting industry. Elements of porous media can be deformed by external force however the mass of each individual element remains unchanged.



Composition	Component sizes $D_k \in [d_k^-, d_k^+]$, $k = 1, \dots, 7$, mm	Content, f_k , $k = 1, \dots, 7$, (mathematical modeling)
1. Coarse-grained sandstone	0.5-1.0	0.04
2. Medium-grained sandstone	0.25-0.5	0.02
3. Fine-grained sandstone	0.1-0.25	0.44
4. Ultrafine-grained sandstone	0.001-0.005	0.17
5. Aleurolitic coarse-grained	0.05-0.1	0.06
6. Fine-grained aleurolitic	0.01-0.05	0.18
7. Pelitic	0.001-0.01	0.09

Applications: Packing soft (deformable, compressible, elastic) objects.

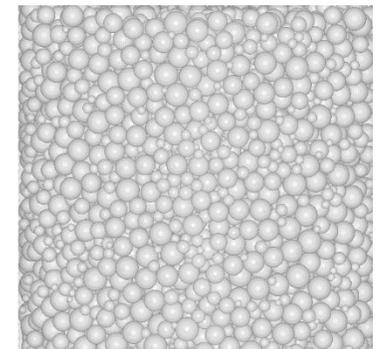
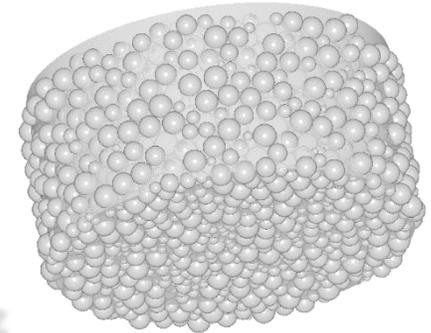


Packing stretched convex polygons in an optimized rectangle, *Wireless Networks*, 2024; Packing Soft Convex Polygons in an Optimized Convex Container, *Mobile Networks and Applications*, 2024.

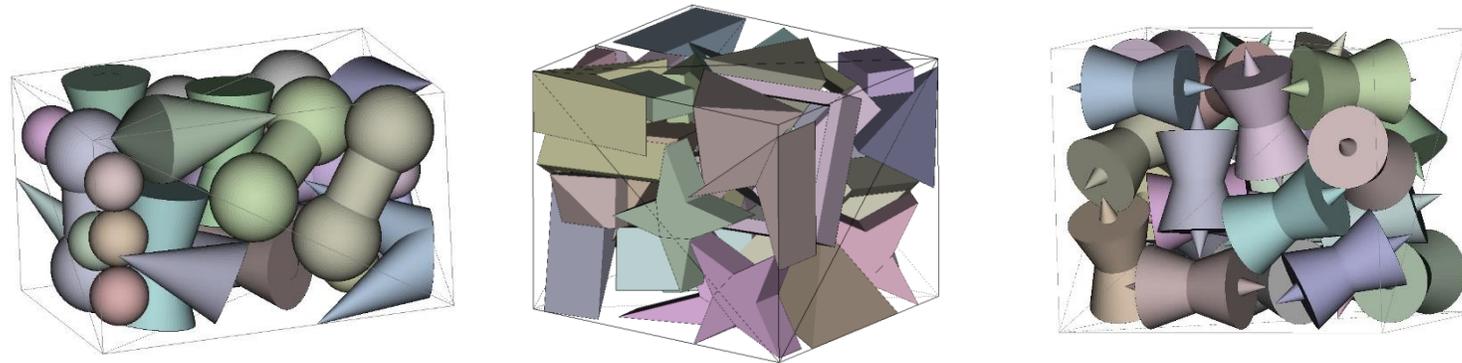


Applications: Mathematical simulation of the material structure in the sample/ Quasi-packing

- A relaxation of classical containment constraints arises in, e.g., analyzing experimentally the porous material by extracting a volumetric sample (container) for further investigation, or by mathematical simulation of the material structure in the sample.
- Allowing a controlled overlap is frequently used in natural sciences since soft spheres can be modelled as hard spheres with a limited overlap. This corresponds to partial overlapping of spheres.

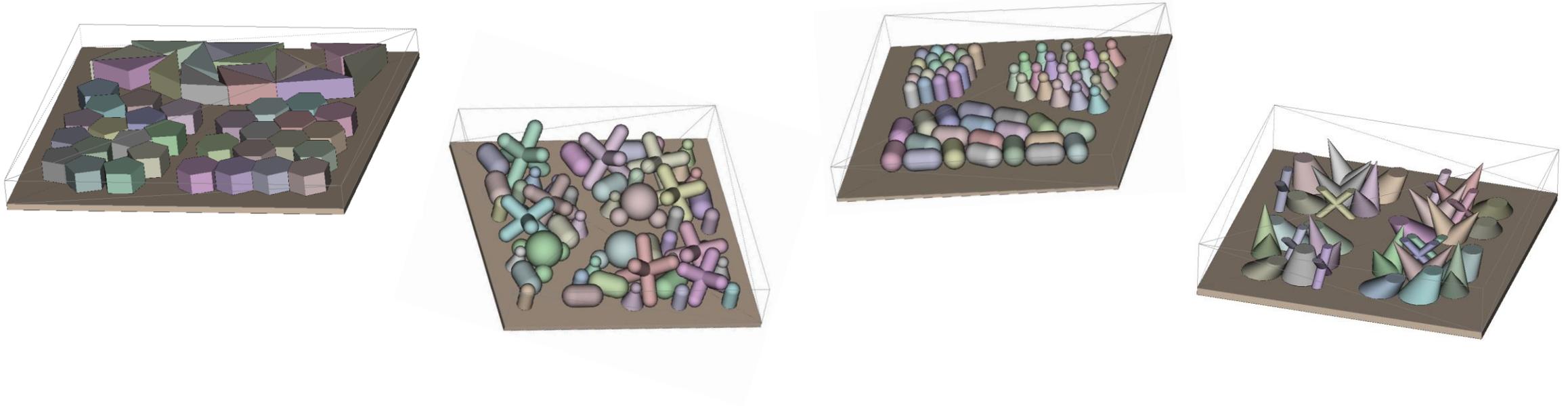


Applications: Packing irregular 3D objects in additive manufacturing

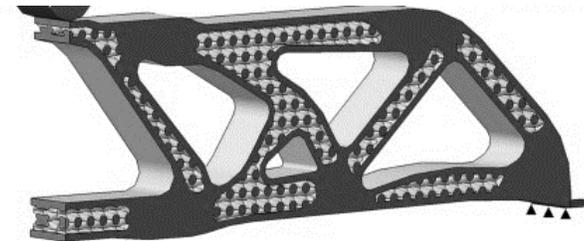
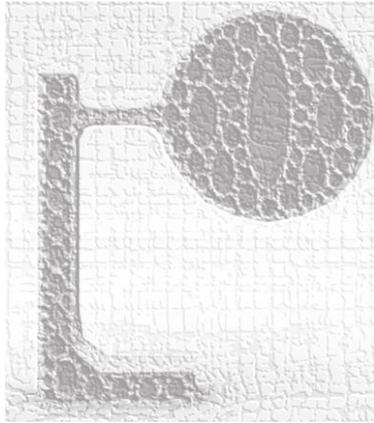


Maximizing the number of 3D parts in a single build volume

Applications: Sparse layout of 3D clusters in AM – Maximizing the number of 3D parts arranged on the building platform regarding 3D printing standards

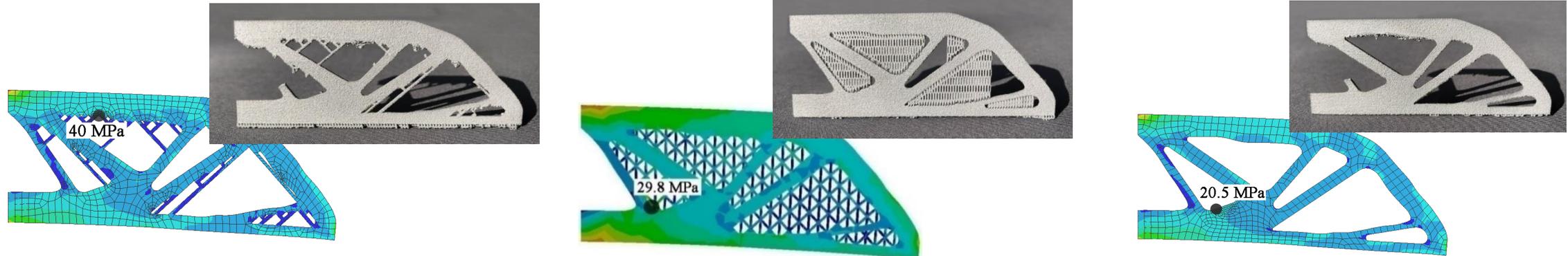


Applications: 3D printing – Generating void structures for topology optimization

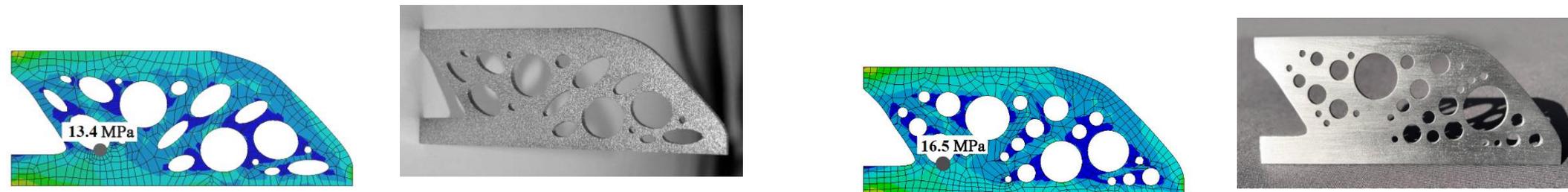


Optimization of the geometry of industrial products in order to save material and improve the quality of 3D printing.

Applications: Topology optimization of parts for 3D printing

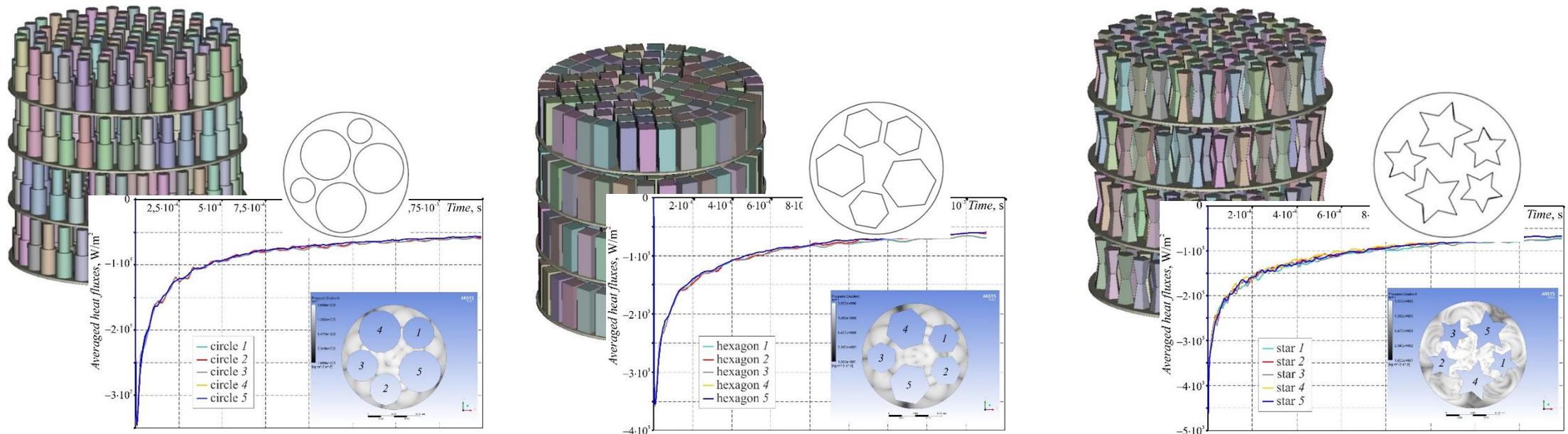


Generating the elliptical/circular void structures: The average level of mechanical stress in the void structures is much smaller than in each of the above models.



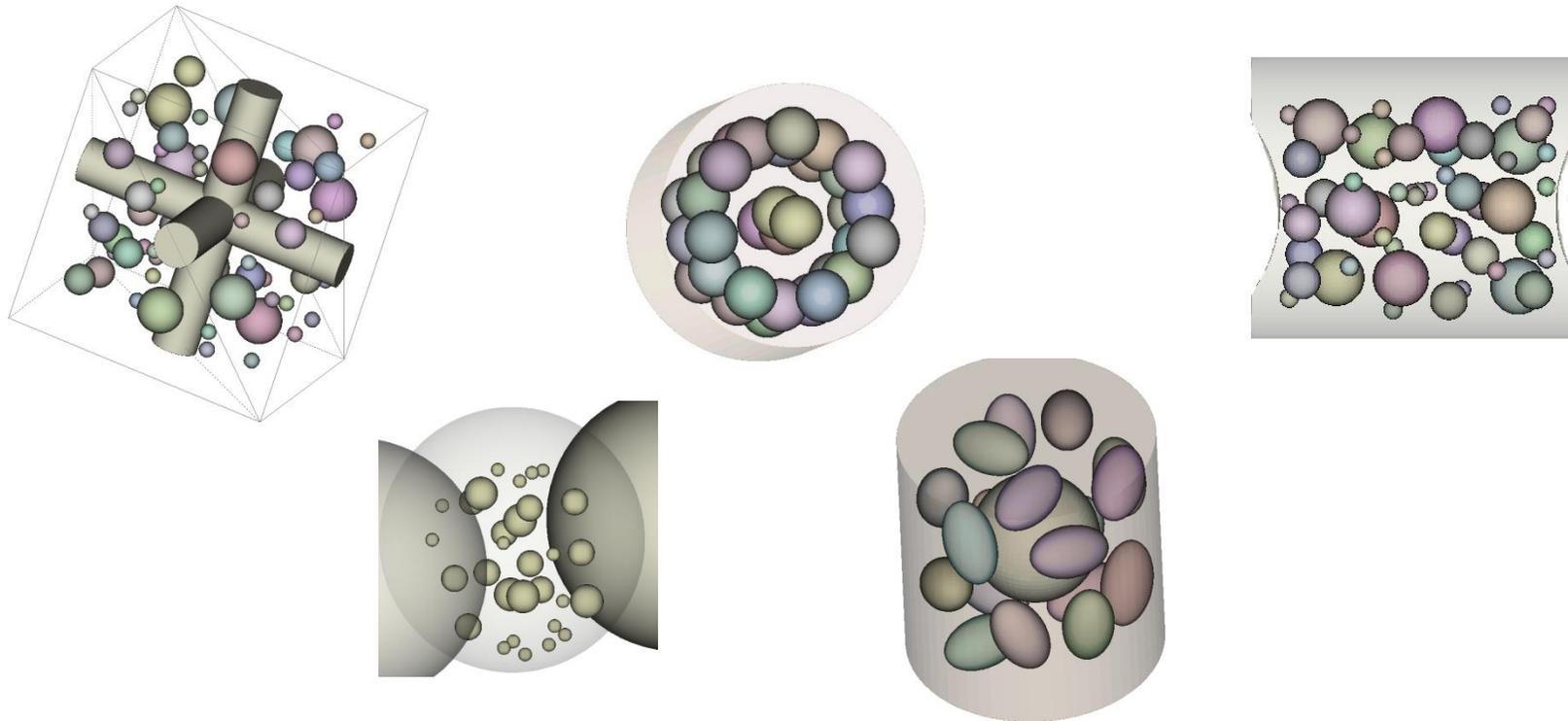
Applications: Sparse packing is used for cleaning 3D parts by TEM

The parts have to be placed sufficiently distant one from another, as well as from the lateral cylindrical surface of the deburring chamber, to achieve the most “uniform” distribution of thermal and power effects.



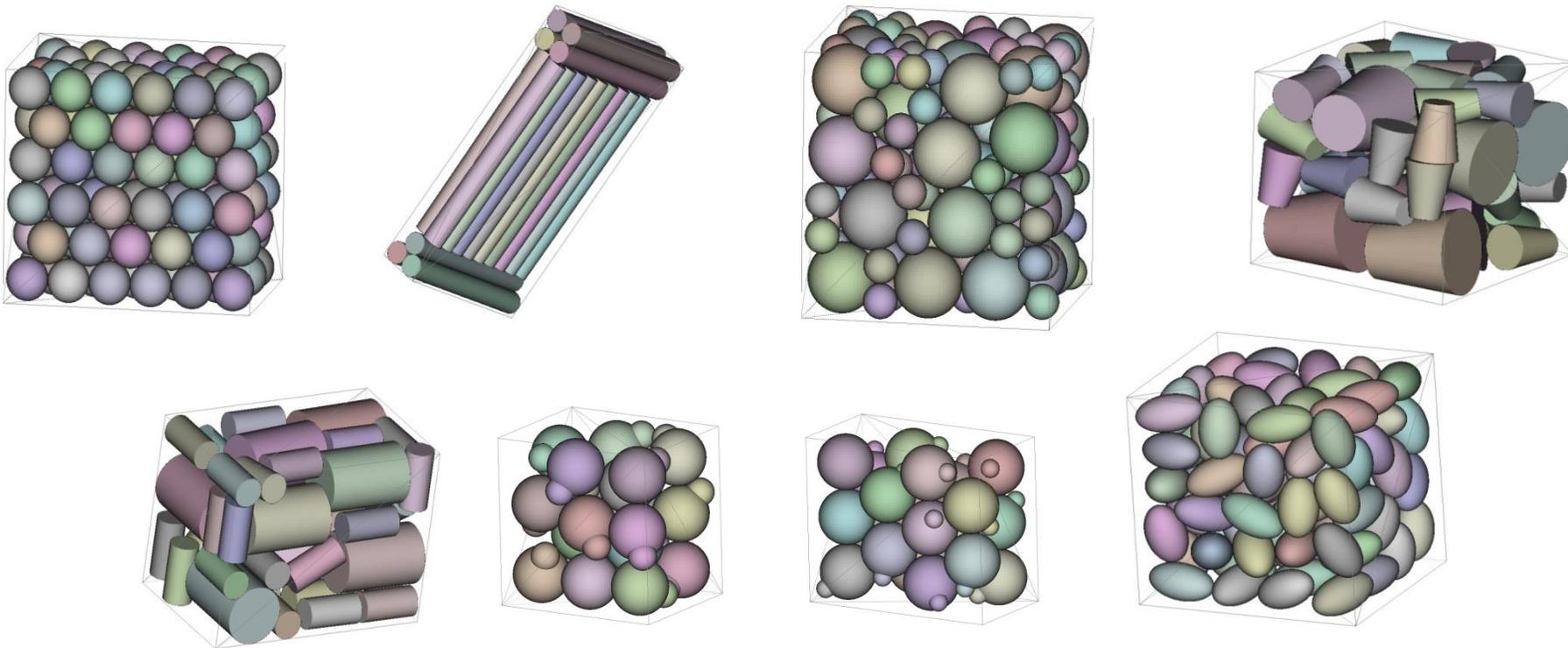
Sparsest packing of two-dimensional objects, *International Journal of Production Research*, 59 (13), 3900-3915, 2021;
Sparse layout of irregular 3D clusters, *Journal of the Operational Research Society*, 74 (1), 351-361, 2023.

Computational results: Sparse Layout for Generating Void Structures in 3D Parts subject to Balance Conditions

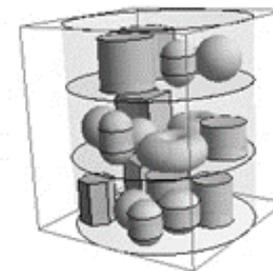
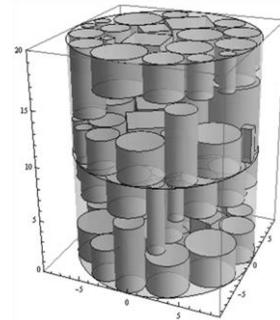
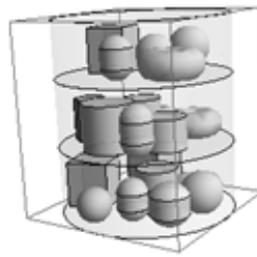
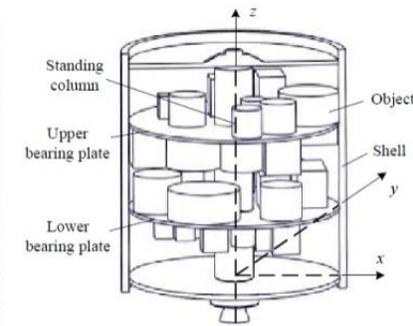
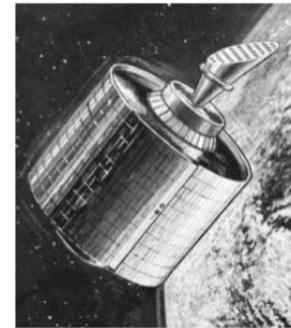
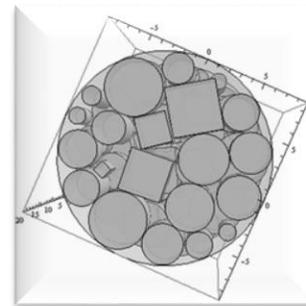
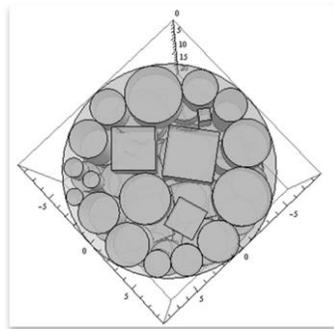


Applications: Filling a given matrix with nano-inclusions

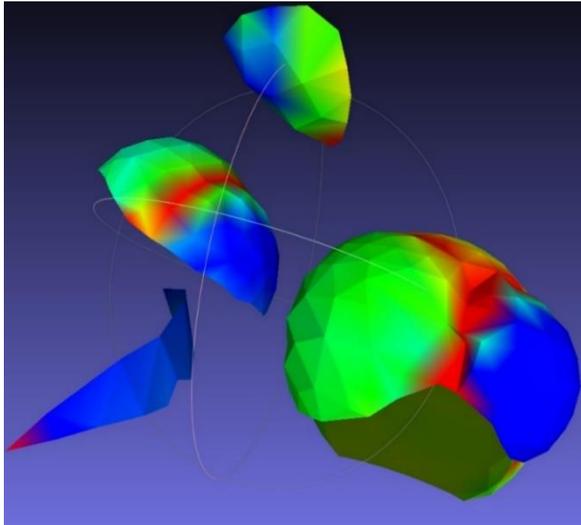
Spherical/satellites, cylindrical, ellipsoidal and conical nano-inclusions are considered in a cuboidal matrix of nanocomposites.



Applications: Balance Layout Problems – an optimized placement of equipment in satellite design (Thales Alenia Space S.p.A, Italy)



Applications: Medicine – Orthopedic Surgery



To restore comminuted fracture when the bone is shattered into small pieces.

Each piece is considered as a part of spherical shell cup (domain) with non-homogeneous density, non-constant curvature and thickness.

The problem consists in placing all the given non-overlapping pieces within the given domain.





Applications: Medicine – Cancer Radiosurgery



A key geometric task in planning treatment with gamma knives is to fill a 3D tumor (irregular domain) with spheres (radiation shots).

The overlapping of spheres can cause an overdose, and a low filling coefficient can cause uneven distribution of the dose of radiation.



Motivation

Development of special geometric tools for modeling placement conditions in packing irregular objects is one of the most challenging problems.

Our objective in this study is to describe a class of irregular packing problems where placement conditions can be stated as simple as in a sphere packing.

The problem of Packing Objects Composed by Generalized Spheres (PCGS) is formulated for objects and containers represented by spheres in arbitrary norms.



Problem Formulation

Let the set of generalized spheres, each with a fixed radius $r_i > 0$ and a variable center $\xi_i \in \mathbb{R}^n$ be defined as follows:

$$S_i(\xi_i) := \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \xi_i\| \leq r_i\}, \quad i \in I := \{1, \dots, m\}$$

The term generalized spheres is used to highlight that by choosing suitable norms, $\|\cdot\|: \mathbb{R}^n \rightarrow [0, \infty]$, a spectrum of spatial shapes can be generated in \mathbb{R}^n .



Useful Norms and Transformations

One of the widely known family of norms is **the Lp norm** that is defined as

$$\|\mathbf{x}\|_p = \left(\sum_k |x_k|^p \right)^{1/p} \text{ for } p \in [1, \infty),$$

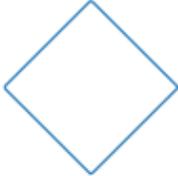
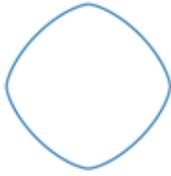
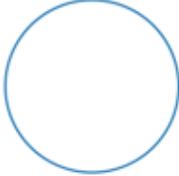
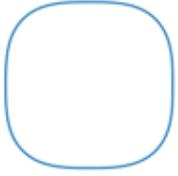
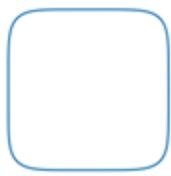
whereas for $p = \infty$

$$\|\mathbf{x}\|_\infty = \max_k |x_k|$$

is the infinity norm.

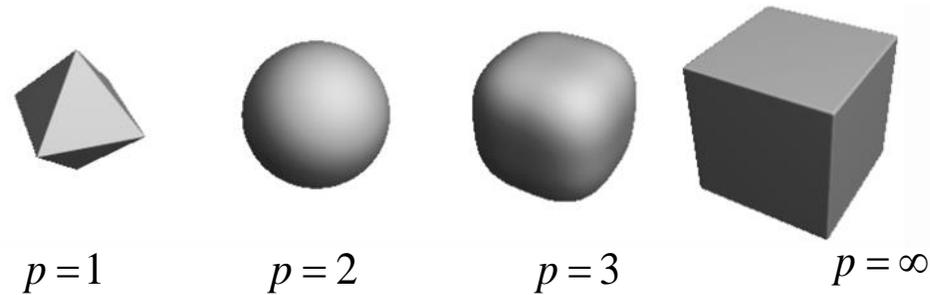


For different $p \in [1, \infty]$, L_p norms generate different convex shapes in \mathbb{R}^2 , e.g., rhombuses, circles, objects with “round” angles, and squares.

		
$p = 1$	$p = 1.5$	$p = 2$
		
$p = 3$	$p = 6$	$p = \infty$

Some shapes of unit spheres in \mathbb{R}^2 for different p

For different $p \in [1, \infty]$, L_p norms generate different convex shapes in \mathbb{R}^3 , e.g., diamonds, spheres, objects with “round” angles, and cubes.



Some shapes of unit spheres in \mathbb{R}^3 for different p



Composition of norms. The maximum of a finite number of norms is also a norm.

In particular, for $\alpha, \beta \in [1, \infty]$,

$$\|\mathbf{x}\|_{comp} = \max\{\|\mathbf{x}\|_{\alpha}, \|\mathbf{x}\|_{\beta}\},$$

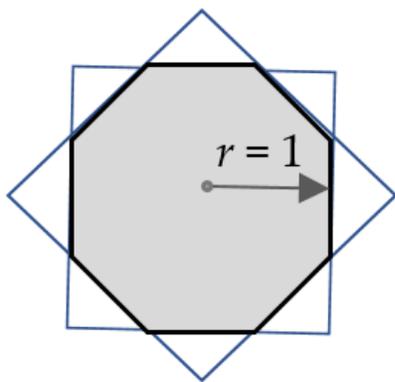
defines a norm.



Example.

$$\begin{aligned}\|\mathbf{x}\|_{comp} &= \max\{\|\mathbf{x}\|_{\infty}, \gamma \|\mathbf{x}\|_1\} \\ &= \max\{|x_1|, |x_2|, \dots, |x_n|, \gamma(|x_1| + |x_2| + \dots + |x_n|)\},\end{aligned}$$

where the first n terms correspond to $\|\mathbf{x}\|_{\infty}$, while the second term with $\gamma > 0$ corresponds to a weighted sum norm $\|\mathbf{x}\|_1$.



For $\mathbf{x} \in \mathbb{R}^2$ and $0.5 < \gamma < 1$, the unit sphere $\|\mathbf{x}\|_{comp} \leq 1$ has an octagon, an intersection of a square and a rhombus (diamond).

Especially for $\gamma = 1/\sqrt{2}$, the regular octagon is obtained.

An octagon as a unit sphere in the composite norm



Problem Formulation: Composed objects

A family of the composed objects $\{A_k \subset \mathbb{R}^n \mid k \in J\}$, $J := \{1, \dots, K\}$ is defined as follows.

Each object A_k is a union of spheres $S_i(\xi_i)$ ($i \in I_k$) $I_k \subset I$ such that

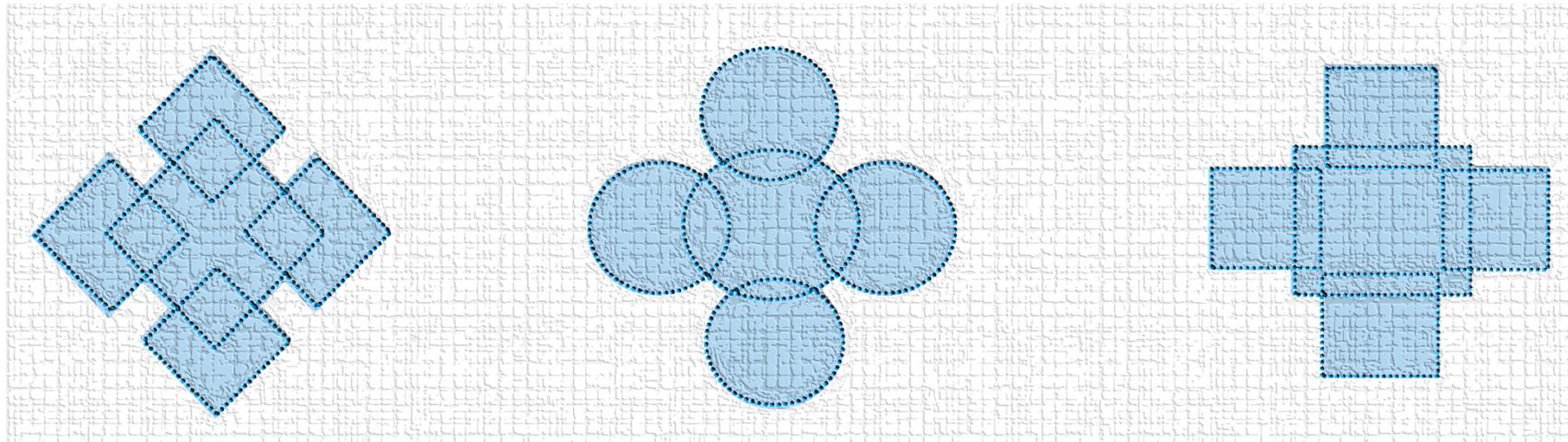
I_1, \dots, I_K is a partition of I , i.e., $I_{k_1} \cap I_{k_2} = \emptyset$ for all $(k_1, k_2) \in J \times J$

with $k_1 \neq k_2$ and $\bigcup_{k \in J} I_k = I$, $I := \{1, \dots, m\}$.

Problem Formulation: Composed objects

$$A_k = \bigcup_{i \in I_k} S_i(\xi_i); \quad (1)$$

$\|\xi_i - \xi_j\| = a_{ij}$ for all $(i, j) \in \mathcal{T}_k := \{(i, j) \in I_k \times I_k \mid i < j\}$ for $k \in J$, $a_{ij} \geq 0$ (composition condition)



Composed object $A = \bigcup_{i=1}^5 S_i(\xi_i)$: (a) $p = 1$, (b) $p = 2$, (c) $p = \infty$.



Problem Formulation

Packing Objects Composed by Generalized Spheres (PCGS) aims to pack a family of objects $A_k, k \in J$ composed by spheres $S_i(\xi_i), i \in I_k$ in a larger sphere $S_0 = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \xi_0\| \leq R\}$ (referred to as a container) to optimize a certain objective function subject to appropriate placement conditions.



Problem Formulation: Placement constraints

The containment and non-overlapping conditions can be written in the form

$$\|\xi_i - \xi_0\| \leq R - r_i, \quad \forall i \in I, \quad (2)$$

$$\|\xi_i - \xi_j\| \geq r_i + r_j, \quad \forall (i, j) \in I_k \times I_l, (k, l) \in \mathcal{K}, \mathcal{K} := \{(k, l) \in J \times J \mid k < l\} \quad (3)$$

Constraint (2) states that each sphere $S_i(\xi_i)$ ($i \in I$) lies completely in the container S_0 . Constraint (3) guarantees that any two spheres from different composed objects do not overlap.



Problem Formulation: Placement constraints

Distance constraints can be stated as follows:

$$\|\xi_i - \xi_0\| \leq R - r_i + \rho_k \quad \forall i \in I_k, k \in J \quad (4)$$

i.e., the distance between the object A_k ($k \in J$) and the boundary of the container S_0 is at least $\rho_k \geq 0$,

$$\|\xi_i - \xi_j\| \geq r_i + r_j + \rho_{kl} \quad \forall (i, j) \in I_k \times I_l, (k, l) \in \mathcal{K}. \quad (5)$$

i.e., the distance between two objects A_k and A_l is at least $\rho_{kl} \geq 0$



A general mathematical model of the PCGS problem can be presented as

$$\min_{(\xi, \tau)} F(\xi, \tau) \text{ subject to } (\xi, \tau) \in G,$$

where $\xi = (\xi_0, \xi_i, i \in I)$ and τ is a vector of auxiliary variables (such as metrical characteristics of the objects or the container). The feasible set G includes elements that satisfy the placement constraints (1)–(3) or (1), (4), (5).

Additional constraints, like balance conditions, restrictions on the values of variables, prohibited spherical zones can be formulated as well.



Some Cases of PCGS Problems

The following three cases of the PCGS problem are considered for $\|\mathbf{x}\|_p$ and $\|\mathbf{x}\|_\infty$: balance, homothetic, and sparse packing for a general case, $\mathbf{x} \in \mathbb{R}^n$.



Generalized **B**alance Packing Problems (GBPP)

In GBPP, a family of the objects must be placed inside a minimum-size container subject to a certain correspondence between the gravity centers of the objects and the container must be ensured.

In addition, a given minimal allowable distance between each pair of objects as well as between each object and the boundary of the container may be given.



GBPP is formulated as a nonlinear programming problem in the form

$$\min_{(\xi, R)} R$$

subject to

$$\|\xi_i - \xi_0\| \leq R - r_i - \rho_k \text{ for } i \in I_k, k \in J,$$

$$\|\xi_i - \xi_j\| \geq r_i + r_j + \rho_{kl} \text{ for } (i, j) \in I_k \times I_l, (k, l) \in \mathcal{K},$$

$$\|\xi_i - \xi_j\| = a_{ij} \text{ for } (i, j) \in \mathcal{J}_k := \{(i, j) \in I_k \times I_k \mid i < j\}$$

$$\sum_i^m w_i \xi_i = \xi_0,$$

where ξ_0 is fixed.

A balance condition guarantees that the gravity center of the system of spheres $S_i(\xi_i)$, $i \in I$ coincides with the center of the container S_0 , where w_i is the weight of $S_i(\xi_i)$, $i \in I$.



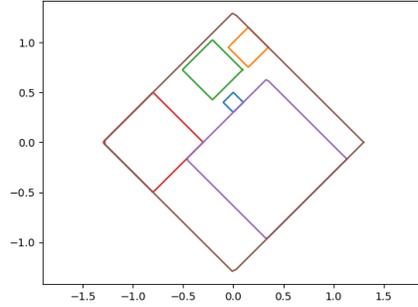
Computational results

Computational results for small instances are provided for the 2D case and various L_p norms to demonstrate the ability of the proposed modeling approach to handle different shapes in a unified way.

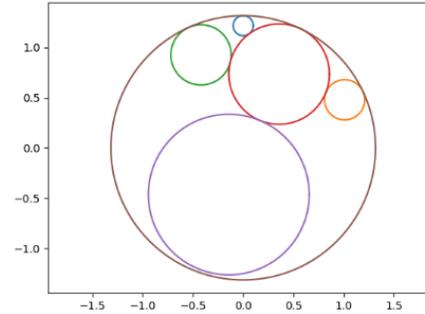
Solutions of NLP problems are obtained by the solver BARON (Tawarmalani and Sahinidis, 2005, Khajavirad and Sahinidis, 2018; Sahinidis, 2021) using the NEOS server and the AMPL platform (Fourer et al., 2003).

We refer to the solution as global if BARON is stopped by fulfilling the optimality criterion, i.e., upper and lower bounds for the objective value coincide.

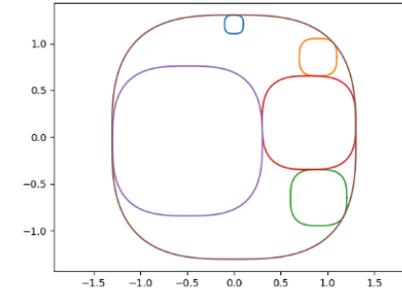
Computational Results for Generalized Balance Packing Problem (GBPP)



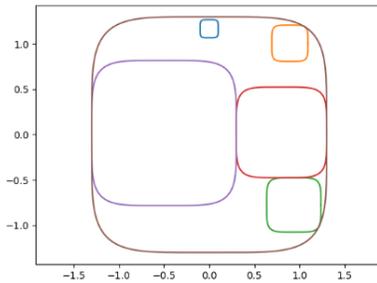
$$p=1, R^* = 1.29999, m=5$$



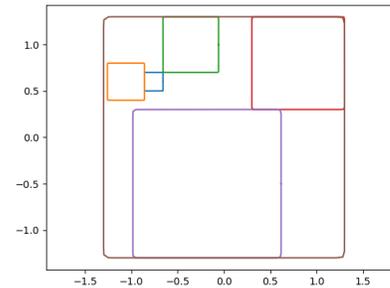
$$p=2, R^* = 1.31624, m=5$$



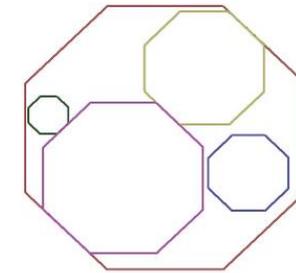
$$p=3, R^* = 1.30375, m=5$$



$$p=6, R^* = 1.3, m=5$$



$$p=\infty, R^* = 1.3, m=5$$

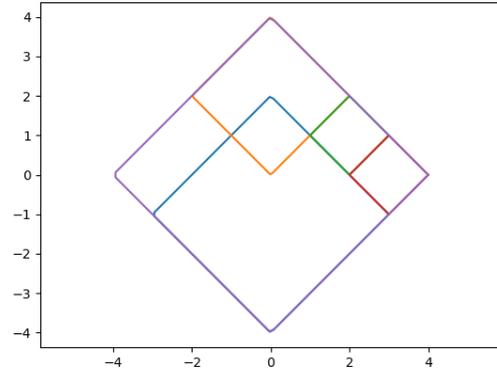


$$R^* = 6.99999, m=4$$

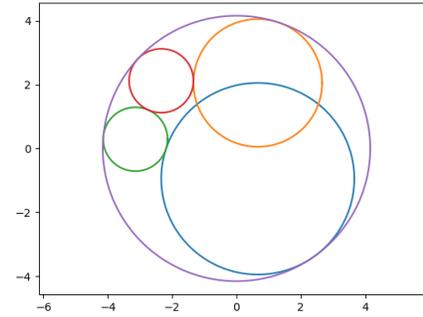
the composite norm

Layouts corresponding to the global solutions obtained by BARON

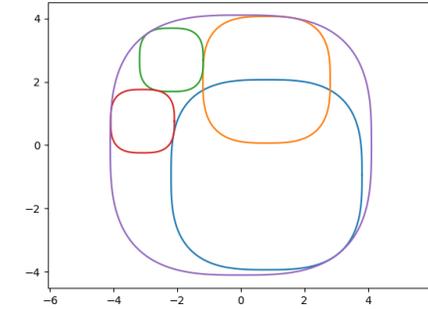
Optimization, Analytics, and Decisions in the Big Data Era
In Honor of the 70-th Birthday of Panos Pardalos
Halkidiki, Greece, June 16-21, 2024



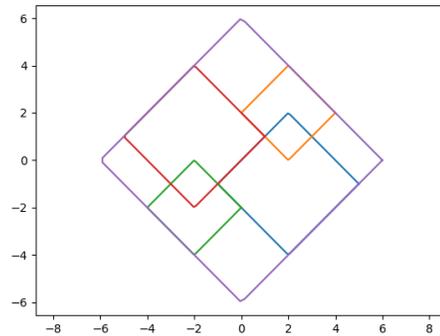
$p=1, R^*=4, m=4, K=3$



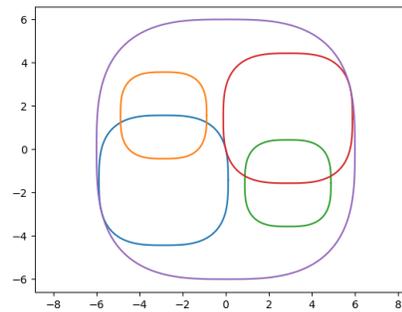
$p=2, R^*=4.15589, m=4, K=3$



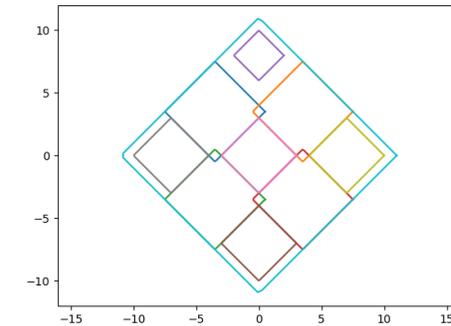
$p=3, R^*=4.10423, m=4, K=3$



$p=1, R^*=6, m=4, K=2$

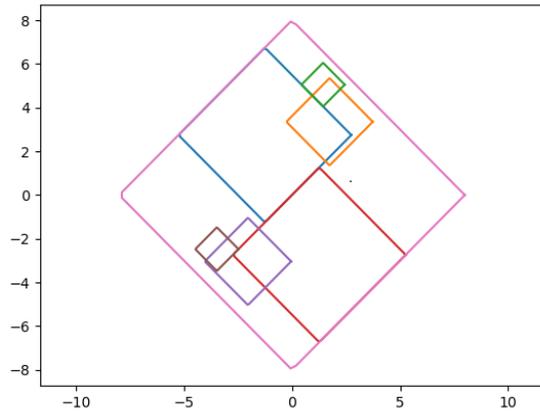


$p=3, R^*=6, m=4, K=2$

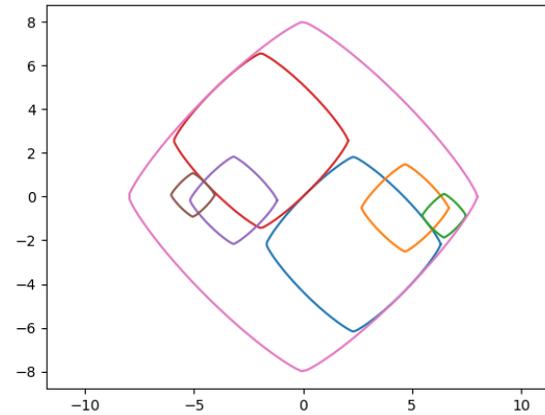


$p=1, R^*=11, m=8, K=5$

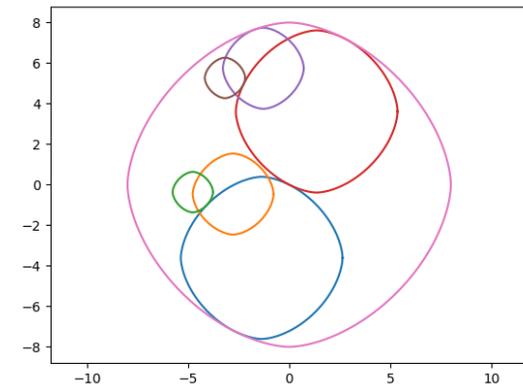
Layouts corresponding to the global solutions obtained by BARON



$p=1, R^*=8, m=6, K=2$



$p=1.2, R^*=8, m=6, K=2$



$p=1.7, R^*=8, m=6, K=2$

Layouts corresponding to the global solutions obtained by BARON



Generalized **Homothetic** Packing Problems (GHPP)

This problem is aimed to pack a family of scaled objects inside a spherical container of a given radius maximizing the scaling parameter.

Consider a family of objects composed by a union of scaled spheres

$$S_i(\lambda, \xi_i) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \xi_i\| \leq \lambda r_i\},$$

where $0 \leq \lambda \leq 1$.



GHPP is formulated as a nonlinear programming problem in the form

$$\max_{(\xi, \lambda)} \lambda$$

subject to

$$\|\xi_i - \xi_0\| \leq R - \lambda r_i \text{ for all } i \in I,$$

$$\|\xi_i - \xi_j\| \geq \lambda(r_i + r_j) \text{ for all } (i, j) \in I_k \times I_l, (k, l) \in \mathcal{K},$$

$$\|\xi_i - \xi_j\| = \lambda a_{ij} \text{ for all } (i, j) \in \mathcal{J}_k := \{(i, j) \in I_k \times I_k \mid i < j\}$$

$$0 \leq \lambda \leq 1,$$

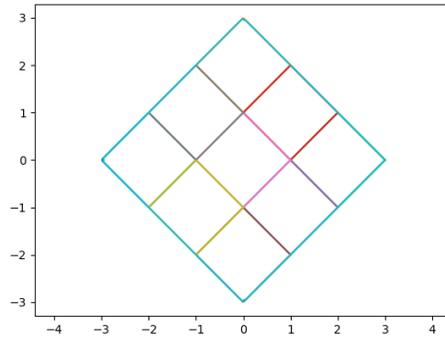
where ξ_0 is fixed.

If $\lambda^* = 1$, then this solution corresponds to a feasible arrangement of full-size spheres $S_i(x_i)$, $i \in I$ inside S_0 .

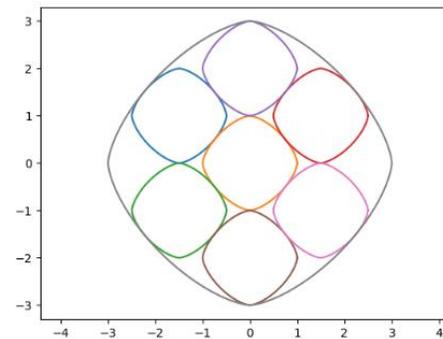
If $\lambda^* < 1$, then it is possible to arrange all m spheres with reduced $\lambda^* r_i$, $i \in I$ inside S_0 .



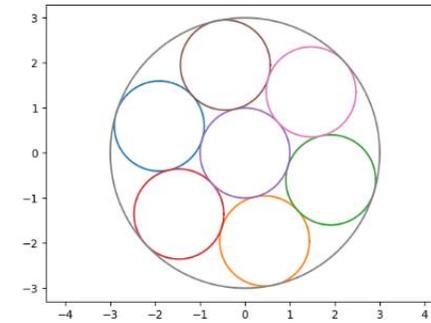
Computational Results for Generalized Homothetic Packing Problem (GHPP)



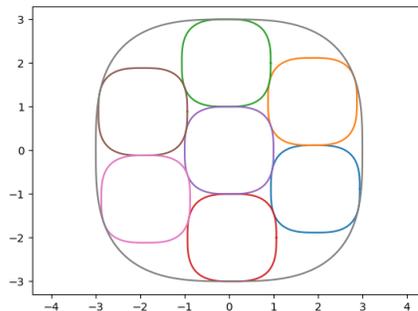
$p=1, \lambda^* = 1, m=9$



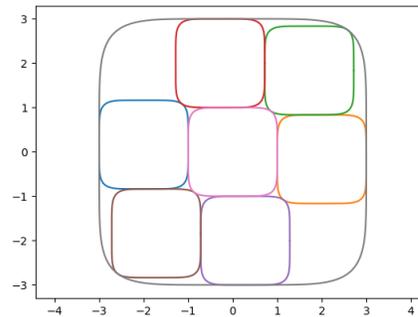
$p=1.5, \lambda^* = 1, m=7$



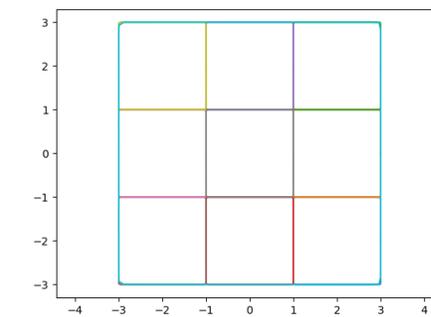
$p=2, \lambda^* = 1, m=7$



$p=3, \lambda^* = 1, m=7$



$p=6, \lambda^* = 1, m=7$



$p=\infty, \lambda^* = 1, m=9$

Layouts corresponding to the global solutions obtained by BARON



Generalized **S**parse Packing Problems (GSPP)

This problem is aimed to pack a family of objects inside a spherical container of a given radius maximizing a minimal distance between each pair of objects as well as between each object and the boundary of the container.



Mathematical model of GSPP with Lp norm.

This model can be given as

$$\max_{(\xi, \rho)} \rho$$

subject to

$$\|\xi_i - \xi_0\| \leq R - r_i - \rho \text{ for } i \in I,$$

$$\|\xi_i - \xi_j\| \geq (r_i + r_j + \rho) \text{ for } (i, j) \in I_k \times I_l, (k, l) \in \mathcal{K},$$

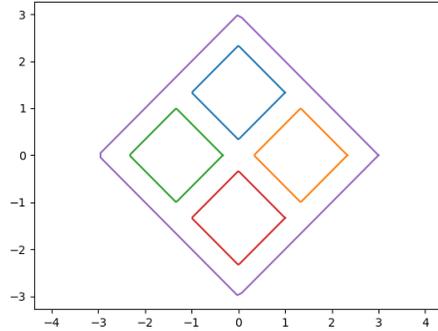
$$\|\xi_i - \xi_j\| = a_{ij} \text{ for } (i, j) \in \mathcal{J}_k := \{(i, j) \in I_k \times I_k \mid i < j\},$$

$$\rho \geq 0,$$

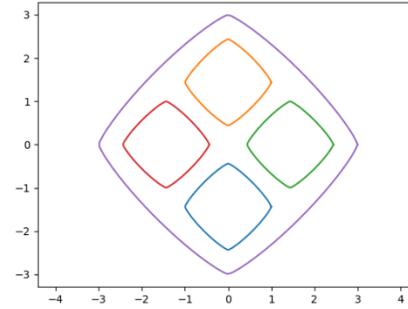
where ξ_0 is fixed.



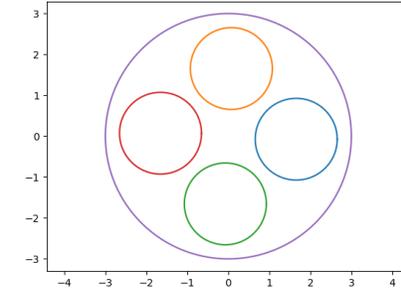
Computational Results for Generalized Sparse Packing Problem (GSPP)



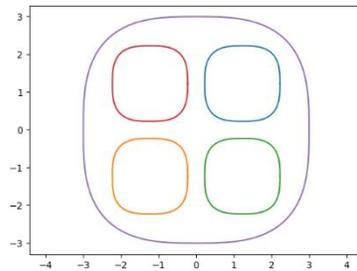
$$p=1, \rho^* = 0.666668, m=4$$



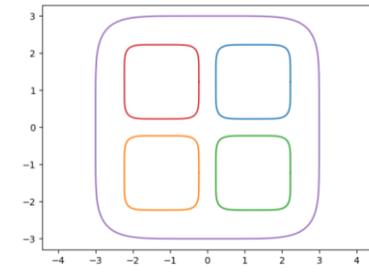
$$p=1.2, \rho^* = 0.562083, m=4$$



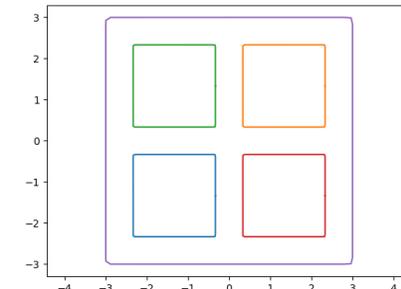
$$p=2, \rho^* = 0.343148, m=4$$



$$p=3, \rho^* = 0.454047, m=4$$



$$p=6, \rho^* = 0.454047, m=4$$

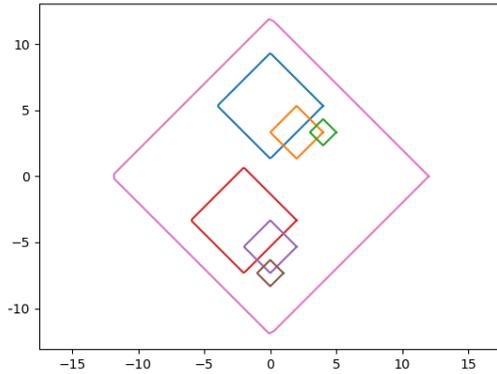


$$p=\infty, \rho^* = 0.666667, m=4$$

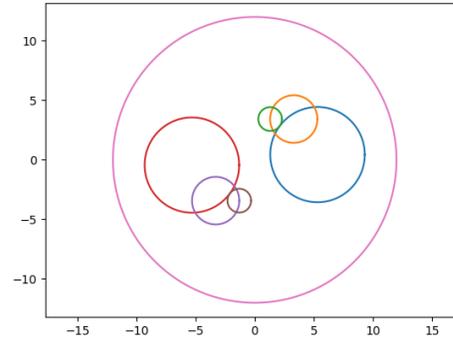
Layouts corresponding to the global solutions found by BARON



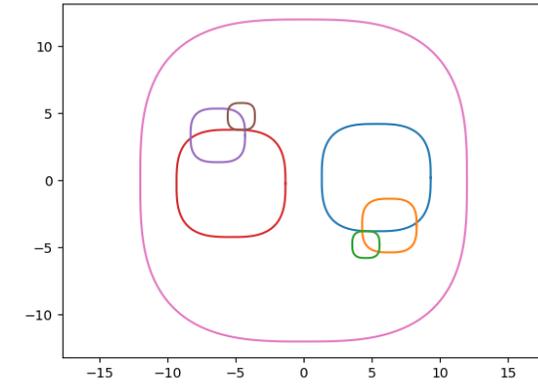
Sparse packing irregular objects composed by spheres defined in L_p norms



$$p=1, \rho^* = 2.66667, m=6, K=2$$



$$p=2, \rho^* = 2.66667, m=6, K=2$$



$$p=3, \rho^* = 2.66667, m=6, K=2$$

LAYOUTS CORRESPONDING TO THE GLOBAL SOLUTIONS FOUND BY BARON



Conclusions

- ✓ A new class of packing problems for n -dimensional objects defined by spheres in terms of an arbitrary norm is introduced and referred to as *Packing Objects Composed by Generalized Spheres (PCGS)*.
- ✓ The main advantage of PCGS is the simplicity of formulating placement conditions for a wide range of regular and irregular shapes.



Future research

- Computational results for larger 2D problem instances.
- Computational results for higher-dimensional problems, $n \geq 3$.
- Using different L_p norms for placement spheres and a container.
- It can be checked: whether conventional exact and heuristic algorithms designed for packing Euclidean spheres are or can be made norm independent.



Acknowledgements

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Optimization, Analytics, and Decisions in the Big Data Era
In Honor of the 70-th Birthday of Panos Pardalos
Halkidiki, Greece, June 16-21, 2024



Thank you
for your attention!