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**THE INDEPENDENCE NUMBER
OF THE GENERALIZED WHEEL GRAPHS W_{2k+1}^p**

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It is known that the maximum independent set problem (MISP) is polynomially solvable for perfect, t -perfect, h -perfect, and W -perfect graphs. The paper discusses the polynomial solvability of MISP for a family of generalized wheel graphs W_{2k+1}^p , which are joints of an odd cycle C_{2k+1} and a clique K_p . We justify that, for any W_{2k+1}^p , the Shor's bound $\psi_1^(G)$ is exact, wherefrom it follows that MISP is polynomially solvable on graphs of this type. The bound $\psi_1^*(G)$ utilizes an equivalent MISP-reformulation as a polynomial optimization problem with two groups of functionally redundant constraints and its dual. It is shown that the polytopes of perfect, t -perfect, h -perfect, and W -perfect graphs are obtainable from the W_{2k+1}^p -graph's polytope by relaxing a part of its constraints whence the polynomial MISP-solvability on these classes of graphs is also followed.*

Keywords: *Maximum Independent Set problem, graph independence number, quadratic optimization, binary optimization, Shor's bound, Lagrange dual bound, polyhedral relaxation, perfect graph*

1. Introduction

Combinatorial Optimization problems are widespread in modeling of real-world problems of Geometric Design and Operational Research. A significant contribution to Combinatorial Optimization and Computational Complexity Theory was made by N. Z. Shor.

He proposed an original technique for obtaining dual bounds on the optimal value of the objective function in quadratic optimization problems [1].

This technique includes an algorithm, called the r-algorithm [2], for obtaining the quadratic dual bounds by solving auxiliary non-smooth continuous optimization problems. Another component of the approach is searching for superflows constraints for the initial problem and then utilizing them to refine the dual bounds. A wide application area of the technique of Shor's dual bounds is Combinatorial Optimization, particularly linear and quadratic binary optimization. Replacement of the binary condition $x \in B_n = \{0,1\}^n$ by quadratic functional dependencies $x_i^2 - x_i = 0, i = \overline{1, n}$ underlies the possibility of solving binary optimization problems as quadratic ones.

In this work, we show the application of Shor's technique of dual bounds for solving the maximum independent set problems (MISPs) on undirected graphs and investigating the complexity of the solutions.

2. Problem statement

Let $G=(V(G),E(G))$ be an undirected graph with vertex set $V(G) = \{1, \dots, n\}$ and edge set $E(G) \subseteq V(G) \times V(G)$.

An *independent set* (stable set) S in G is a subset of $V(G)$, whose elements are pairwise nonadjacent.

Let $S(G)$ be a set of all independent sets in G . Then $\forall S \in S(G), \forall i, j \in S, i \neq j$: the condition $\{i, j\} \notin E(G)$ holds.

The maximum independent set problem (MISP) is to find an independent set $S^* \in S(G)$ of maximum cardinality. A maximal independent set (MIS) is an independent set that is not a subset of any other independent set, i.e., MISP is a problem of finding an MIS S^* . The size of a maximum independent set is called *the stability number* of G and is denoted by $\alpha(G)$. Thus the MISP consists in finding the stability number.

MISP can be modelled as follows: find

$$\alpha(G) = |S^*| = \max_{S \in S(G)} |S|.$$

2.1 The polytope $STAB(G)$ and its relaxations

$$STAB(G) = \text{conv}\{\chi(S) : S \in S(G)\}$$

is a polytope of incidence vectors of independent sets in G (here, $\chi(S)$ is an incident vector of S). Some valid inequalities of $STAB(G)$ are

$$\text{(vertex constraints)} : 0 \leq x_i \leq 1, i = \overline{1, n};$$

$$\text{(edge constraints)} : x_i + x_j \leq 1, (i, j) \in E(G);$$

$$\text{(clique constraint)} : \sum_{i \in V(Q)} x_i \leq 1, \forall Q \in Q(G);$$

$$\text{(odd-cycle constraints)} : \sum_{i \in V(C_{2k+1})} x_i \leq k, \forall C_{2k+1} \in C(G),$$

$$\text{(odd-wheel constraints)} : \sum_{i \in V(C_{2k+1})} x_i + kx_{i_{2k+2}} \leq k, W_{2k+1}^1 \in W^1(G),$$

where $Q(G), C(G), W^1(G)$ are sets of cliques, odd cycles and wheels in G .

Taken in different combinations with vertex constraints, these constraints induces a relaxation polytope $LSTAB(G)$, which coincides with $STAB(G)$ for a certain family of graphs called L -perfect. Among them are perfect, t -perfect, h -perfect, odd-wheel perfect graphs. Identity $LSTAB(G) = STAB(G)$ reduces solution of a MISP to its linear relaxation

over $STAB(G)$, but three latter groups of constraints are, generally, are of an exponential size. Nevertheless, a MISP is polynomially solvable by polynomial optimization tools, particularly, by Shor's dual bound approach. It utilizes quadratic formulations of MISP, having the form of

$$\text{(quadratic vertex constraints): } x_i^2 - x_i = 0, i = \overline{1, n};$$

$$\text{(quadratic edge constraints): } x_i x_j = 0, (i, j) \in E(G);$$

$$\text{(superflows constraints): } x_i x_k + x_j x_k \leq x_k,$$

$$k = \overline{1, n}, (i, j) \in E(G).$$

We extend a set of known so far L -perfect graphs on which MISP is polynomially solvable by extending wheel constrains to generalized odd-wheel constraints:

$$\sum_{i \in V(C_{2k+1})} x_i + k \sum_{j \in V(Q_p)} x_j \leq k, W_{2k+1}^p \in W(G),$$

where $k, p \in \mathbb{N}$, $W(G)$ is a set of generalizes odd wheels that are join of an odd cycle and a clique. Justification of the polynomial solvability is based on utilization of derived discrete optimization MISP-formulations. These new MISP-statements were obtained by applying a technique if continuous functional representation of combinatorial sets.

Summary

This work is dedicated to applying Shor's technique of quadratic dual bounds to MIS problems and deriving classes of polynomially solvable MISPs.

In particular, the paper substantiates the polynomial solvability of MISPs on odd wheel-perfect graphs W_{2k+1}^1 and their generalizations, such as generalized odd-wheel perfect graphs W_{2k+1}^p .

The theory of continuous functional representations of discrete sets was applied in designing a variety of different formulations of MISP. These results underly the theoretic justification of the polynomial solvability of MISP for some classes of graphs. In such a way, we provide a new proof of

MISP polynomial solvability for perfect, t -perfect, h -perfect graphs and substantiate this for the first time for a family of W_{2k+1}^p -perfect graphs. In particular, with the help of functional representations' techniques, all known so far continuous, binary and partially binary mathematical models were constructed, and several new models were obtained. Based on this, new families of superflows constraints of the polytope $STAB(G)$ of incidence vectors of independent sets in a graph G were also derived. Expectedly, these families of constraints will allow proving a polynomial solvability of other classes of graphs, in particular, superclasses of W_{2k+1}^p -perfect graphs.

The presented technique of obtaining new mathematical formulations of COPs and the models themselves has theoretical value. In the future, the results can be applied in computation experiments to identify more efficient models in certain cases of input data.

References

1. Shor, Naum Z. (1998). Nondifferentiable optimization and polynomial problems. Dordrecht: Kluwer Academic Publishers.
2. Shor, Naum Z., and Stetsyuk, Petro I. (1997). "The use of a modification of the r-algorithm for finding the global minimum of polynomial functions," *Cybern Syst Anal*, no. 4, pp. 28–49.