

THE SOLVING METHOD OF MULTICRITERIA LINEAR OPTIMIZATION PROBLEM IN INTEGERS

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Abstract

A wide range of practical optimization problems in various fields lead to the solution of multicriteria linear optimization models [1] in integers. Into the current paper we propose a method for solving the multicriteria model of linear type in integers of interactive type. Thus, the decision maker, initially assigning a certain utility to each criterion, will finally build a uni-criterion model of linear optimization in integers. The imposition of each criterion quantified in the synthesis function remains at the discretion of the decision maker, the optimal values and weight being calculated in whole or real numbers, which does not change the optimal solution of the model. To this end, the decision-maker has at his disposal a selection of combinatorial values of the objective functions, which depends on the number of criteria in the initial model.

The theoretical justification of the algorithm is brought in the paper. The algorithm was tested on several examples, which proved its veracity.

Keywords: Multi-criteria model in integers, efficient solution, optimal compromise solution.

Subject classifications: 90C10, 90C27, 90C29

1. Introduction

The major importance of using mathematical optimization in integers is due to the need to obtain integer solutions in various modeled practical situations. Among the practical fields of application of the solution of the optimization model in integers, a special place belongs to the problem of

one, two and three-dimensional cutting [1], [4]. A number of studies can be listed here, such as: dynamic memory allocation, solving problems on multiprocessor systems and general positioning problems (Coffman et al. 1978, Garey and Johnson 1981, Coffman and Leighton 1989, Dyckhoff 1990). The two-dimensional variant of the cutting problem is of NP complexity due to its combinatorial explosion with increasing size of the problem (Garey and Johnson 1979). The exact methods were investigated by Gilmore and Gomory (1961) and are considered the first methods actually applied in the tailoring industry. Recently, Cung and other researchers (2000) developed an algorithm, which allows the exact solution of some variants of two-dimensional cutting problems. But when the problem is of multicriteria type, even linear, this effort is further amplified. That said, the condition that the decision variables belong to the set of integers creates a major difficulty, the problem gaining another level of complexity and is solved in a longer time [1], [4]. The scientific research study for this field remains open [2], given that there is a wide range of multicriteria models of fractional linear type, fuzzy, etc., which for application reasons must be solved in whole numbers.

2. Problem formulation

The integer multicriteria linear optimization problem is usually described by a set of linear constraints, such as equations and / or inequalities, including on the variables constraints of non-negativity and integrity. The decisional problem with an infinite number of variants is described as follows:

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} \min \\ \max \end{array} \right\} F_k(x) = \sum_{j=1}^n c_{kj} x_j, \quad k = \overline{1, r} \\ A \cdot x \leq b \\ x \in Z^+ \end{array} \right. \quad (1)$$

where : $A = \|a_{ij}\|$ is an array of size $m \times n$ ($m < n$), $C = \|c_{kj}\|$, is an array of size $r \times n$ ($r < n$), x is a vector n -dimensional column, and b is a m -dimensional column vector.

The interpretations of the parameters c_{kj} may be the most different, according of their practical meanings such as unit costs or benefits, or others close in meaning. Their significance determines the type of the corresponding objective function, minimum or maximum. Analogously, the elements of the matrix A , a_{ij} , represent the specific consumption of the resource j for the production of a product unit of type i , and the elements of the vector b represent the available by types of resources.

We note that in model (1) it is possible to have some criteria of minimum type and others of maximum type, for example, maximizing benefits, profit or others or minimizing costs, depreciation, loss or others.

3. Theoretical landmarks

In order to solve the multicriteria optimization model in integers (1), we will propose some analogous approaches to those in real numbers.

1. The solution $x^* \in Z^+$ is the vector that optimizes a synthesis function of r objective functions, ie: $h(F) = h[F_1, F_2, \dots, F_r]$, in which $h(\cdot)$ it can be defined in several, various ways [5].

2. The solution $x^* \in Z^+$ is the vector which minimizes one criterion in the form: $\phi(x^*) = \min_{x \in D} h(\psi_1(x - X_1), \dots, \psi_r(x - X_r))$,

in which, $X_j = (x_{1j}, x_{2j}, \dots, x_{nj})^T$, $j = \overline{1, r}$ is the optimal solution to the problem with a single objective function, F_j , and ψ_k is a distance type function between vector $x \in D$ and optimal solution X_k for the corresponding criterion F_k .

4. The solution $x^* \in Z^+$ is the vector which belongs to a set of effective whole-type points.

Because the model (1) is of multi-criteria type, it's known that such kind of model rarely admits the optimal solutions in integers.

Definition 1 The basic solution X^* of the model (2), where $X^* \in Z^+$ is called optimal overall if it is the optimal solution for each of criteria.

By solving model (1) we will assume the construction of a finite set of its efficient integer solutions known again as a Pareto-optimal or non-dominated solutions [5], solutions of the best compromise. We will further propose the definition of the efficient solution for the multicriteria linear deterministic problem in integers.

Definition 2 The basic solution \bar{X} , where $\bar{X} \in Z^+$ of the model (2) is an basic efficient one if and only if it doesn't exists any other basic solution $X \in Z^+$, where $X \neq \bar{X}$, which would improve the values of all criteria and at least one criterion would be strictly improved.

We propose the same definition in a more rigorous form.

Definition 3 The basic solution X^* of the model (1), where $X^* \in Z^+$ is one of the optimal(best) compromises solution if it is located closed to the optimal solutions of each criterion.

4. Combinatorial synthesis algorithm for solving the linear multicriteria optimization

One of the most important problems that arises when solving the multicriteria optimization problem in integers using the methods of synthesis functions is: what kind of optimal solutions of each criterion we will use to build the synthesis function of all criteria, these being in R^+ or in Z^+ , so that the final model solve it in Z^+ ? In this justified paragraph we will answer this question.

In order to solve the multicriteria model of linear optimization in integers of type (1) we will apply the method of synthesis functions,

namely we will use the method of maximizing global utility, which we will achieve in two stages.

Stage I

1. At this stage we will solve $2r$ unicriteria linear programming problem from model (1) of type: $F_j = \underset{x \in D}{\text{optim}} F_j(x)$ and

$F_j^p = \underset{x \in D}{\text{pessim}} F_j(x)$, on the admissible domain:

$$D = \{x \in R \mid Ax \leq b, x \geq 0\};$$

2. Next we will solve $2r$ more linear programming problems of the type: $F_j = \underset{x \in D}{\text{optim}} F_j(x)$ and $F_j^p = \underset{x \in D}{\text{pessim}} F_j(x)$, on the admissible

domain:

$$D = \{x \in Z^+ \mid Ax \leq b, x \geq 0\};$$

3. We will combinatorial select the vectors of optimal values and corresponding to the pessimistic values of the objective functions, some calculated on Z^+ , others in R^+ . The number of such combinations is finite because the size of the problem is finite. These can be described as follows:

$$\left\{ \left(\begin{matrix} F_1(R^+) \\ F_2(R^+) \\ \dots \\ F_r(R^+) \end{matrix} \right) \vee \left(\begin{matrix} F_1(R^+) \\ F_2(Z^+) \\ \dots \\ F_r(Z^+) \end{matrix} \right) \vee \left(\begin{matrix} F_1(R^+) \\ F_2(R^+) \\ \dots \\ F_r(Z^+) \end{matrix} \right) \vee \dots \vee \left(\begin{matrix} F_1(Z^+) \\ F_2(Z^+) \\ \dots \\ F_r(Z^+) \end{matrix} \right) \right\},$$

$$\left\{ \left(\begin{matrix} F_1^p(R^+) \\ F_2^p(R^+) \\ \dots \\ F_r^p(R^+) \end{matrix} \right) \vee \left(\begin{matrix} F_1^p(R^+) \\ F_2^p(Z^+) \\ \dots \\ F_r^p(Z^+) \end{matrix} \right) \vee \left(\begin{matrix} F_1^p(R^+) \\ F_2^p(R^+) \\ \dots \\ F_r^p(Z^+) \end{matrix} \right) \vee \dots \vee \left(\begin{matrix} F_1^p(Z^+) \\ F_2^p(Z^+) \\ \dots \\ F_r^p(Z^+) \end{matrix} \right) \right\}$$

The number of such vectors is: $N(V) = C_r^1 + C_r^2 + \dots + C_r^r$, the same as the number of vectors with pessimistic value records of the criteria.

Stage II

1. By selecting one of the vector records of the values of the objective functions and the vector of the corresponding records of the pessim values, we will construct the synthesis function, which expresses the summary utility of the criteria: $G = \sum_{j=1}^r (\alpha_j F_j + \beta_j)$, which must be maximized. The coefficients $\{(\alpha_j, \beta_j)\}_{j=1, r}$ are determined by applying the global utility maximization algorithm, described above.

2. We will determine the optimal solution of the next model:

$$\max_{x \in D} G = \sum_{j=1}^r (\alpha_j F_j(X) + \beta_j), \text{ where: } D = \{x / A \cdot x = b, x \in Z^+\}, \text{ that is}$$

the optimal compromise solution for model (2). Either that is it X^* . We will calculate the values of each objective function in this solution and we

will build the next vector of records of the criteria:
$$\left\{ \begin{matrix} F_1(X^*) \\ F_2(X^*) \\ \dots \\ F_r(X^*) \end{matrix} \right\}.$$

Theorem. For a set of a priori utilities assigned to the criteria in model (1), the solution of the optimal compromise of the integer model remains the same for any vector of the optimal records of the combinatorial criteria calculated in R^+ or in Z^+ .

Proof. Let X_{eff}^1 be a solution of the optimal compromise for the whole type model (1), which records the smallest distance to the optimal whole type solutions of each criterion. We will assume that the synthesis function of the final model was constructed using a combination of

optimal values of the objective functions from model (1), some being solved in R^+ , others in Z^+ .

$$\text{Let: } \begin{pmatrix} F_1(R^+) \\ F_2(R^+) \\ \dots \\ F_r(Z^+) \end{pmatrix} \text{ -vector of the optimal and pessimim } \begin{pmatrix} F_1^p(R^+) \\ F_2^p(R^+) \\ \dots \\ F_r^p(R^+) \end{pmatrix} \text{ recorded}$$

values of objective functions.

We will assume that for another recording values of the objective functions from the model (1), different from the previous one, let it be:

$$\begin{pmatrix} F_1(Z^+) \\ F_2(R^+) \\ \dots \\ F_r(Z^+) \end{pmatrix}, \text{ and corresponding vector of the pessimim values } \begin{pmatrix} F_1^p(Z^+) \\ F_2^p(R^+) \\ \dots \\ F_r^p(Z^+) \end{pmatrix},$$

the objective synthesis function registered another solution of the optimal compromise in integers, different from the first, either it is X_{eff}^2 . If

$X_{eff}^1 \neq X_{eff}^2$, then there is at least one coordinate after which these vectors

differ. Therefore, at least for one criterion, let it be with indexes i_1 , the distance between its optimal solution in integers and the new solution is smaller than the previous one, ie the relationship is fair:

$\rho(X_{eff}^1, X_{i_1}^*) > \rho(X_{eff}^2, X_{i_1}^*)$, where $X_{i_1}^*$ is optimal solution in integer of

criterion i_1 , which contradicts the assumption that X_{eff}^1 is the solution of the

optimal compromise in integers for the model (1), which had to be demonstrated. So, our assumption is wrong. Therefore, the model (1) admits

a single solution of the optimal compromise in integers, regardless of the

configuration of records of the optimal values of the criteria in R^+ or Z^+ ,

used in the construction of the synthesis function of the model.

Remark 1. For any vector of combinatorial records of the values of the objective functions of the unicriteria models of the problem (1) in R^+ or in Z^+ , and for their utilities, the optimal compromise solution of the model (1) in integers remains the same.

Remark 2. For any new set of initial utilities assigned to the criteria in model (1), applying the method of maximizing the maximum utility we will obtain a new solution of the optimal compromise in integers for all the criteria of this model.

5. Conclusions

The proposed paper brings an efficient algorithm in solving the linear multicriteria optimization model in integers. We focused on the use of the methods of synthesis functions, namely the method of maximizing the global utility in solving the multicriteria model of linear type in integers, which leads us to determine an optimal compromise solution, closest to the optimal solutions in integers of each separate criterion. To determine this, the decision maker can use both the optimal value of each criterion in integers and in real numbers, both positive. The set of all possible combinations of such vectors for recording the values of the objective functions as well as the weight values was exploited. Regardless of the configuration used to construct the synthesis function, its optimal solution in integers does not change. So, the decision maker can select the most advantageous values for calculating the objective functions - synthesis, which is a very important moment that certainly increases the efficiency of the algorithm.

Example

For the following linear model of multicriteria optimization in integers find the solution of the optimal compromise using the method of synthesis functions, for the proposed utilities of criteria.

$$\min \{F_1(X) = x_1 + 2x_2 + x_3\}$$

$$\max \{F_1(X) = 2x_1 + x_2 + 2x_3\}$$

$$\max \{F_1(X) = 2x_1 + 3x_2 + x_3\}$$

$$\begin{cases} 3x_1 + 5x_2 + x_3 \leq 18 \\ 5x_1 + 3x_2 + 2x_3 \leq 20 \\ 2x_1 + x_2 + 2x_3 \geq 5 \\ x_j \in \mathbb{Z}^+ \end{cases}$$

F_1	F_2	F_3	F_1^p	F_2^p	F_3^p
$U_1 = 4$	$U_2 = 8$	$U_3 = 9$	$U_1 = 1$	$U_2 = 2$	$U_3 = 2$

Solving procedure:

Using the proposed algorithm we obtained eight synthesis functions according of table of proposed utilities for the model criteria. These are the next:

$$F_1(U) = 1,73x_1 + 1,63x_2 + 1,09x_3 \rightarrow \max$$

$$F_2(U) = 1,83x_1 + 1,75x_2 + 1,13x_3 \rightarrow \max$$

$$F_3(U) = 1,85x_1 + 1,8x_2 + 1,15x_3 \rightarrow \max$$

$$F_4(U) = 1,85x_1 + 1,8x_2 + 1,15x_3 \rightarrow \max$$

$$F_5(U) = 1,73x_1 + 1,63x_2 + 1,09x_3 \rightarrow \max$$

$$F_6(U) = 1,7x_1 + 1,57x_2 + 1,07x_3 \rightarrow \max$$

$$F_7(U) = 1,83x_1 + 1,75x_2 + 1,13x_3 \rightarrow \max$$

$$F_8(U) = 1,7x_1 + 1,57x_2 + 1,07x_3 \rightarrow \max$$

Solving in turn these 8 problems of linear programming in integers in the same admissible domain, we obtained the following solutions of the optimal compromise:

$$X_{eff}^1 =$$

$$X_{eff}^2 = X_{eff}^3 = X_{eff}^4 = X_{eff}^5 = X_{eff}^6 = X_{eff}^7 = X_{eff}^8 = X^* = \{x_1^* = 1, x_2^* = 3, x_3^* = 0\}$$

We calculated the values of the utility functions, which are the following:

$$F_1(U) \approx 11,89; \quad F_2(U) \approx 12,105; \quad F_3(U) \approx 12,02; \quad F_4(U) \approx 12,02; \\ F_5(U) \approx 11,89; \\ F_6(U) \approx 11,93; \quad F_7(U) \approx 12,105; \quad F_8(U) \approx 11,93; \text{ and obtained the}$$

$$\text{next values of each criterion: } = \left\{ \begin{array}{c} 7 \\ 5 \\ 11 \end{array} \right\};$$

References

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