ON STABILITY OF MULTICRITERIA PARAMETRIZED INVESTMENT PROBLEM WITH SAVAGE'S RISK CRITERIA

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Annotation. *On the basis of the portfolio theory, a multicriteria investment Boolean problem of minimizing lost profits with parameterized efficiency is formulated. The problem considered is the finding a set of all efficient portfolios. The quality of such portfolios is assessed by examining stability of the set of efficient portfolios to perturbations of minimax risk criterion parameters. The lower and upper bounds of the stability radius are obtained in the case of arbitrary Hölder's norms being specified in the three spaces of the problem initial data.*
Keywords*: Multicriteria optimization, Savage's risk criteria, set of efficient portfolios, stability radius, Hölder's metric, investment problem..*

In optimization a question of stability of a problem arises in the case where the set of feasible solutions and (or) the choice function depend on parameters, for which the area of change is known only. The presence of such parameters in optimization models is caused by inaccuracy of the initial data, non-adequacy of models to real processes, errors of numerical methods, errors of rounding off and other factors. Hence it appears important to allocate classes of problems in which small changes of input data lead to small changes of the result. The problems with such properties are called stable. It is obvious that many optimization problem cannot be correctly formulated and solved without use of results of the stability theory [1].

In the current work we develop the concept of a quantitative study of the stability [2] of a problem and deal with a quantitative measure of the level of data perturbation that does not violate efficiency, known as the stability radius. We research stability aspects of the multiobjective investment problem [3] for the case of the so-called parametrized efficiency ”from extreme to Pareto” and provide lower and upper bounds on the stability radius.

Consider a multicriteria discrete variant of the investment optimization problem with the following parameters specified below.

Let $N_n = \{1,2,\ldots,n\}$ be a variety of alternatives (investment assets); N_m be a set of possible financial market states (market situations, scenarios); N_s be a set of possible risks; r_{ijk} be a numerical measure of economic risk of type $k \in N_s$ if investor chooses project $j \in N_n$ given the market is in state $i \in N_m$; $R = \left[r_{ijk} \right] \in \mathbf{R}^{m\times n\times s}$ be a matrix specifying risks; $x = (x_1, x_2, \ldots, x_n)^T \in \mathbf{E}^n$ be an investment portfolio, where $\mathbf{E} = \{0,1\}$, and $x_j = 1$ if investor chooses project j, otherwise $x_j = 0$; $X \subset \mathbf{E}^n$ be a set of all admissible investment portfolios; $\mathbf{R}^m$ be a financial market state space; $\mathbf{R}^n$ be a portfolio space; $\mathbf{R}^s$ be a risk space.

Efficiency of a chosen portfolio (Boolean vector) $x \in X, |X| \geq 2$ is evaluated by a vector objective function

$$f(x,R) = \left(f(x,R_1), f(x,R_2), \ldots, f(x,R_s) \right)^T,$$

with each partial objective representing minimax Savage's risk criterion

$$f(x,R_k) = \max_{i\in N_m} r_{ik}x = \max_{i\in N_m} \sum_{j\in N_n} \mathrm{r}_{ijk} x_j \to \min_{x\in X}, \; k \in N_s,$$

where $r_{ik} = (r_{i1k}, r_{i2k}, \ldots, r_{ink}) \in \mathbf{R}^n$, $i \in N_m$, $k \in N_s$, $R_k \in \mathbf{R}^{m\times n}$ represents the k-th cut of the risk matrix $R = \left[r_{ijk} \right] \in \mathbf{R}^{m\times n\times s}$ with rows r_{ik}.

For arbitrary $v \in \mathrm{N}$, we define the Pareto dominance between two vectors $y = (y_1, y_2, \ldots, y_v)^T \in \mathbf{R}^v$ and $y' = (y'_1, y'_2, \ldots, y'_v)^T \in \mathbf{R}^v$:

$$y \succ y' \Leftrightarrow y \geq y' \; \& \; y \neq y'.$$

Next, let $\varnothing \neq I \subseteq N_s, |I| = h, \; I = \{k_1, k_2, \ldots, k_h\}, \; 1 \leq k_1 < \ldots < k_h \leq s$, $R_I = \left(R_{k_1}, R_{k_2}, \ldots, R_{k_h} \right)^T \in \mathbf{R}^{m\times n\times h}$; for any $x \in X$ $f(x,R_I) = = \left(f(x,R_{k_1}), f(x,R_{k_2}), \ldots, f(x,R_{k_h}) \right)^T$; $u \in N_s$ and $N_s = \bigcup_{v\in N_u} I_v$ be a partition of the set N_s, where $I_v \neq \varnothing$, $v \in N_u$, and $i \neq j \Rightarrow I_i \cap I_j = \varnothing$. For the given partition, we introduce a set of $(I_1, I_2, \ldots, I_u)$-efficient portfolios according to the following formula:

$$G_m^s(R, I_1, I_2, \ldots, I_u) = G_m^{su}(R) = \left\{ x \in X : \ \exists v \in N_u \ \left(X\left(x, R_{I_v}\right) = \varnothing \right) \right\},$$

where $X\left(x, R_{I_v}\right) = \left\{ x' \in X : f\left(x, R_{I_v}\right) \succ f\left(x', R_{I_v}\right) \right\}.$ It is easy to see that the set of efficient portfolio is non-empty.

In one particular case, if $u = 1$, i.e. $I = N_s$, the set $G_m^s(R, N_s)$ is Pareto set $P_m^s(R) = \left\{ x \in X : X(x, R) = \varnothing \right\},$ where $X(x, R) =$
$= \left\{ x' \in X : f(x, R) \geq f(x', R) \ \& \ f(x, R) \neq f(x', R) \right\}.$

In another particular case, if $u = s$, i.e. $I_v = \{v\}$ for $v \in N_u = N_s,$ the set $G_m^s\left(R, \{1\}, \{2\}, \ldots, \{s\}\right)$ is a set of all the so-called extreme portfolios $E_m^s(R) = \left\{ x \in X : \ \exists k \in N_s \ \left(X(x, R_k) = \varnothing \right) \right\},$
where $X(x, R_k) = \left\{ x' \in X : f(x, R_k) > f(x', R_k) \right\}$.

The problem of finding the set of efficient portfolios $G_m^{su}(R)$ is referred to as multicriteria investment Boolean problem with Savage's risk criteria of different types and denoted by $Z_m^s(R, I_1, I_2, \ldots, I_u)$, or shortly, $Z_m^{su}(R)$.

In the spaces $\mathbf{R}^n, \mathbf{R}^m$ and $\mathbf{R}^s$ we define three Hölder's norms l_p, l_q and $l_t,$ where $p, q, t \in [1, \infty]$. Recall, that Hölder's norm l_p of a vector $a = (a_1, a_2, ..., a_n)^T \in \mathbf{R}^n$ is the number

$$\| a \|_p = \begin{cases} \left(\sum_{j \in N_n} | a_j |^p \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\left\{ | a_j | : j \in N_n \right\} & \text{if } p = \infty. \end{cases}$$

So, the norm of matrix $R \in \mathbf{R}^{m \times n \times s}$ is the number $\| R \|_{pqt} = \left\| \left(\| R_1 \|_{pq}, \| R_2 \|_{pq}, ..., \| R_s \|_{pq} \right) \right\|_t$ with cuts
$\| R_k \|_{pq} = \left\| \left(\| r_{1k} \|_p, \| r_{2k} \|_p, ..., \| r_{mk} \|_p \right) \right\|_q, \quad k \in N_s.$ Following [1], the stability radius of $Z_m^{su}(R),\ s, m \in \mathbf{N},$ is defined as $\rho_m^{su}(p,q,t) = \sup {}_{pqt},$ if

$\Xi_{pqt} \neq \varnothing$, and $\rho_m^{su}(p,q,t)=0,$ if $\Xi_{pqt} = \varnothing.$ Here $\Xi_{pqt} = \left\{\varepsilon > 0 \ : \forall R' \in \Omega_{pqt}(\varepsilon) \ \left(G_m^{su}(R+R') \subseteq G_m^{su}(R)\right)\right\}$,

$\Omega_{pqt}(\varepsilon) = \left\{R' \in \mathbf{R}^{m\times n\times s} \ : \ \| R' \|_{pqt} < \varepsilon\right\}$, $k \in N_s$.

Obviously, if $G_m^{su}(R) = X$, then the stability radius is not bounded. The problem $Z_m^{su}(R)$ with $X \setminus E_s^m(R) \neq \varnothing$ is called *non-trivial*.

For non-trivial problem $Z_m^{su}(R)$, we notice:

$$\varphi = \min_{v\in N_u} \min_{x \notin G_m^{su}(R)} \max_{x' \in P\left(x, R_{I_v}\right)} \min_{k \in I_v} \frac{f(x,R_k) - f(x',R_k)}{\left\| \left(\| x \|_{p^*}, \| x' \|_{p^*} \right) \right\|_{\gamma}},$$

$$\psi = n^{\frac{1}{p}} m^{\frac{1}{q}} \min_{v\in N_u} \min_{x \notin G_m^{su}(R)} \max_{x' \in P\left(x, R_{I_v}\right)} \min_{k \in I_v} \frac{f(x,R_k) - f(x',R_k)}{\| x - x' \|_1} |I_v|^{\frac{1}{t}},$$

$$\sigma = \sigma^s(p,q) = \min\left\{\| R_k \|_{pq} : \ k \in N_s\right\}, \ \gamma = \min\left\{p^*, q^*\right\}.$$

$$P\left(x, R_{I_v}\right) = P\left(R_{I_v}\right) \cap X\left(x, R_{I_v}\right), \ \ P\left(R_{I_v}\right) = \left\{x \in X : X\left(x, R_{I_v}\right) = \varnothing\right\}.$$

It is easy to see that $\varphi, \psi \geq 0$.

Theorem. *For any* $s, m \in \boldsymbol{N}, \ \ u \in N_s$, *and* $p, q, t \in [1, \infty]$, *the stability radius* $\rho_m^{su}(p,q,t)$ *of* s *-criteria non-trivial problem* $Z_m^{su}(R)$ *has the following lower and upper bounds* $\varphi \leq \rho_m^{su}(p,q,t) \leq \min\{\psi, \sigma\}$.

References.

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