FUNCTIONAL LIMIT THEOREMS FOR PERTURBED RANDOM WALKS

V. BOGDANSKII,
A. PILIPENKO
Igor Sikorskii Kyiv Polytechnic Institute, Kyiv, Ukraine
vbogdanskii@ukr.net
pilipenko.ay@gmail.com

Abstract. *Consider Markov chain on Z with jumps of two types: outside of a finite set* A *the jumps are mean zero i.i.d. random variables with finite variance, jumps from* A *may be arbitrary. We study the Donsker scaling limist of such Markov chains. Depending on properties of jumps from* A *we obtain various limit processes such as a Brownian motion with reflection, a Brownian motion with semipermeable membrane, a Brownian motion with jump type exit from 0, etc. Multidimensional generalizations are given*
Key words: *Perturbed random walks, diffusions with semipermeable membrane, functional limit theorems.*

Let $\{\xi_n\}$ be a sequence of mean-zero i.i.d. random variables with finite second moment. Set $S_n = \xi_1 + ... + \xi_n$. It is well known that the Donsker scaling of the sequence $S_{[nt]}/\sqrt{n}$ converges as $n \to \infty$ in distribution to a Brownian motion.

Let A be a fixed set. Consider the following perturbation of the random walk $\{\tilde{S}_n\}$ at the set A. Put $\tilde{S}_{n+1} := \tilde{S}_n + \xi_{n+1}$ if $\tilde{S}_n \notin A$ and $\tilde{S}_{n+1} := \tilde{S}_n + \eta_{i,n+1}$ if $\tilde{S}_n = i \in A$, where $\{\eta_{i,n}\}_{n\ge1}$ are independent sequences of independent identical distributed random variables.

If A is a finite set, then Donsker's scaling limits may be interpreted as a diffusion with semipermeable membrane [1]. If $A = (-\infty, 0] \cap Z$ and $\{\eta_{i,n}\}_{n\ge1}$ are non-negative then corresponding models appear in queuing theory. In particular, if $\eta_{i,n} = -i$, then $\{\tilde{S}_n\}$ is the classical Lindley recursion [2,3].

Further we will assume that all states of the perturbed walk $\{\tilde{S}_n\}$ are connected.

Theorem 1. *Assume that $\eta_{i,n}$ are non-negative and their distribution is independent of i.*

(a) If $E\eta_{i,n} < \infty$, then $\{\tilde{S}_{[nt]} / \sqrt{n}\}$ converges in distribution to the reflected Brownian motion.

(b) If tail of $\eta_{i,n}$ distribution is a slowly varying functions, then $\{\tilde{S}_{[nt]} / \sqrt{n}\}$ converges to $+\infty$ in probability.

(c) If tail of $\eta_{i,n}$ distribution is a regularly varying function with index $\alpha \in (0,1)$, then $\{\tilde{S}_{[nt]} / \sigma\sqrt{n}\}$ converges in distribution to the process

$$W_\alpha(t) = W(t) + U_\alpha \circ U_\alpha^{(-1)} \circ M(t),$$

where $\{W(t)\}$ is a Brownian motion, $\{U_\alpha(t)\}$ is a drift-free α-stable subordinator independent of W, $M(t) = -\min_{s\in[0,t]} W(s)$.

The proof of Theorem 1 is based on the generalized Skorokhod reflection principle [4].

The following theorem gives the form of the limit process if the membrane is finite but the jumps from the membrane have both signs.

Theorem 2. *Assume that random variables $\{\xi_n\}$ are bounded and the set A is finite. Then $\{\tilde{S}_{[nt]} / \sqrt{n}\}$ converges in distribution to the skew Brownian motion, i.e., continuous Markov process with transition probability density function*

$$p_t(x,y) = \varphi_t(x-y) + \gamma sign(y)\varphi_t(|x|+|y|),$$

where φ_t is the density of the normal distribution $N(0,t)$.

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