

Two families of convex problems for finding capacities of arcs of a fault-tolerant network

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We present mathematical models, methods and software for two classes of problems for finding capacities of arcs of a fault-tolerant oriented network (a fault-tolerant network is a network for which all requirements for transmission of flows can be satisfied both in the case of absence of faults in the network and if one arbitrary fault from all possible faults in the network occurs). In the first class of problems (**problem A**), all possible paths in the network can be used to transfer flows. In the second class of problems (**problem B**), only paths from a predetermined set are used for transferring flows. In other words, in problem A, flow transmission is determined by the physical structure of a network, while in problem B, it is determined by its logical structure. Mathematical models are represented by nonlinear convex programming problems with the network structure of the constraint matrix. The developed algorithms and software are based on implementations of subgradient methods with space transformation [1] and take into account structural features of optimization models [2, 3].

Keywords: fault-tolerant networks, the network physical structure, the network logical structure, convex programming, subgradient methods with space transformation, Shor's r-algorithm.

Introduction

The world in the 21st century is becoming increasingly dependent on reliability of various kinds of communications. It is natural that the problems of finding both structures and parameters of stable and reliable networks (telecommunications, computer, transport, energy, etc.) are

becoming more and more relevant. We consider some types of "faults" of arcs in an oriented network. Their essence will be explained in the example of road networks.

Let y_{ij} be capacity of an arc, which corresponds to a section of a road from vertex i to vertex j where is a three-row (three-lane) car traffic. If one lane of this section of the road turns out to be blocked, then the capacity y'_{ij} of the arc (i, j) can be represented as

$$Y'_{ij} = \frac{2}{3} y_{ij}$$

that is, the arc (i, j) operates on $2/3$ ($\mu = 2/3$) of its full capacity. If two lanes are blocked, then

$$Y'_{ij} = \frac{1}{3} y_{ij},$$

which corresponds to the operation of the arc (i, j) with a capacity of no more than $1/3$ ($\mu = 1/3$) of its maximum capacity. Similar situations occur in telecommunication networks if information is transmitted over a given number of lines (cables), some of which can either fail or switch for serving alternative flows.

Fault-tolerant networks

Using the parameter μ , various fault scenarios, including reducing capacities of vertices in an oriented network, can be described. This possibility is the basis of convex programming problems described below modeling fault-tolerant telecommunication, transportation, and other networks.

Definition. *Let define a fault of arc (i, j) , as the change of its capacity according to the following rule:*

$$Y'_{ij} = \mu y_{ij} \quad \mu \in [0,1).$$

If $\mu = 0$, then $y'_{ij} = 0$ and this is equivalent to the complete failure of the arc (i, j) .

Let $N = (V, A)$ be an oriented network with a set of vertices V and a

set of arcs A . The capacity of arc $a = (i, j) \in A$ is denoted by y_a .

Let T be the set of faults (fault scenario) for the network $N = (V, A)$. Each fault $t \in T$ is determined by capacities of arcs, where

$$Y_a^t = \mu_{at} y_a, \quad \mu_{at} \in [0,1], \quad \forall a \in A, \quad \forall t \in T.$$

In Table 1 the faults scenario 0.5F is given for the network $Net(6,8)$ (consisting of 6 vertices and 8 arcs, see Fig. 1), where one arbitrary arc reduces its capacity by half ($\mu = 0.5$). If for scenario 0.5F all 0.5 values are replaced by zero, then we get a new faults scenario 1F, equivalent to a complete failure of only one arbitrary arc in the network $Net(6,8)$.

TABLE I
SCENARIOS 0F (WITHOUT FAULTS) AND 0.5F (ONE
ARBITRARY ARC REDUCES ITS CAPACITY BY HALF)

Arcs		0F	0,5F (one arbitrary arc, $\mu = 0,5$)							
#	(i, j)	μ_0	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8
1	(1,2)	1,0	0,5	1,0	1,0	1,0	1,0	1,0	1,0	1,0
2	(1,3)	1,0	1,0	0,5	1,0	1,0	1,0	1,0	1,0	1,0
3	(2,4)	1,0	1,0	1,0	0,5	1,0	1,0	1,0	1,0	1,0
4	(2,5)	1,0	1,0	1,0	1,0	0,5	1,0	1,0	1,0	1,0
5	(3,4)	1,0	1,0	1,0	1,0	1,0	0,5	1,0	1,0	1,0
6	(3,5)	1,0	1,0	1,0	1,0	1,0	1,0	0,5	1,0	1,0
7	(4,6)	1,0	1,0	1,0	1,0	1,0	1,0	1,0	0,5	1,0
8	(5,6)	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	0,5

Let us note that faults of vertices are not considered since they can be reduced to faults of arcs in a "new" network.

The essence of problems of finding capacities of arcs in a fault-tolerant network is as follows.

Given:

1. A network $N = (V, A)$; the value y_a^0 of the existing capacity of an arc $a \in A$.

2. Commodities K ; d_k , $k \in K$ is demand of commodity k from node $s(k) \in V$ to node $r(k) \in V$; the node $s(k) \in V$ is called a source (a

sender) and $r(k) \in V$ is called a sink (a receiver).

3. A faults scenario T for the network $N = (V, A)$.

We need to find: optimal (by some criterion) values of capacities y_a^* , $a \in A$ (added to the existing ones y_a^0 , $a \in A$), which provides a given amount of traffic K in the network $N = (V, A)$ when any fault from the faults scenario T occurs.

For the network $N = (V, A)$, we will consider two types of problems of finding arc capacities.

Problem A: for transfer of flows all possible paths in the network are used.

Problem B: for transfer of flows only paths from a given set of paths P are used. Here $P = \cup_{k \in K} P_k$, where P_k is the set of paths for the flow k .

In other words, the transfer of flows in problem A is determined by the physical structure of the network, and in the problem B it is determined by the logical (certain set of possible paths) network structure. Examples of both structures are shown in Fig. 1 and Fig. 2.

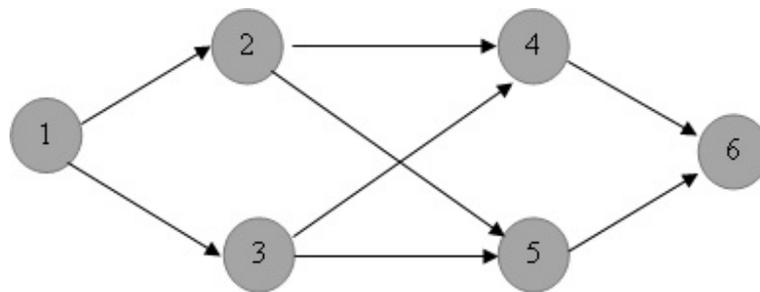


Fig 1. The physical structure of the network $Net(6,8)$

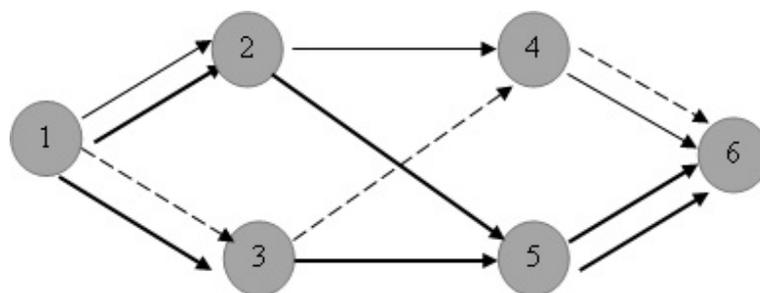


Fig 2. The logical structure of the network $Net(6,8)$

Convex optimization problems A and B

Let $f_A(Y)$ be a convex function of the capacities of arcs $Y = \{y_a, a \in A\}$, A_i^+ and A_i^- be sets of arcs entering and exiting from a vertex $i \in V$. Then the problem A has the following form:

$$f_A^* = f_A(Y^*) = \min_{X,Y} f_A(Y) \quad (1)$$

subject to:

$$\sum_{k \in K} x_{akt} \leq \mu_{at}(y_a^0 + y_a), \quad \forall a \in A, \forall t \in T, \quad (2)$$

$$\sum_{a \in A_i^+} x_{akt} - \sum_{a \in A_i^-} x_{akt} = \begin{cases} d_k, & \text{if } i = s(k); \quad \forall i \in V, \\ -d_k, & \text{if } i = r(k); \quad \forall k \in K, \\ 0 & \text{otherwise; } \quad \forall t \in T, \end{cases} \quad (3)$$

$$x_{akt} \geq 0, \quad \forall a \in A, \forall k \in K, \forall t \in T, \quad (4)$$

$$y_a^{low} \leq y_a \leq y_a^{up}, \quad \forall a \in A. \quad (5)$$

Here, the variable x_{akt} denotes the flow of commodity k along arc a if fault t occurs; $s(k) = \text{sender}(k)$; $r(k) = \text{receiver}(k)$.

The formulation of the problem B is as follows:

$$f_B^* = f_B(Y^*) = \min_{Z,Y} f_B(Y) \quad (6)$$

subject to:

$$\sum_{k \in K} \sum_{p \in P_k} \delta_{kpa} z_{kpt} \leq \mu_{at} (y_a^0 + y_a), \quad \forall a \in A, \forall t \in T, \quad (7)$$

$$\sum_{p \in P_k} z_{kpt} = d_k \quad \forall k \in K, \forall t \in T, \quad (8)$$

$$z_{kpt} \geq 0, \quad \forall k \in K, \forall p \in P, \forall t \in T, \quad (9)$$

$$y_a^{low} \leq y_a \leq y_a^{up}, \quad \forall a \in A, \quad (10)$$

where

$$\delta_{kpa} = \begin{cases} 1, & \text{if path } p \in P_k \text{ includes arc } a; \\ 0 & \text{otherwise.} \end{cases}$$

Here, $f_B(Y)$ is a convex function, and the variable z_{kpt} denotes the flow of commodity $k \in K$ along the path $p \in P_k$ if fault $t \in T$ occurs.

The essence of the constraints in problems A and B is the following: constraints (2) and (7) means that the flows along the arcs should not exceed the capacities of the arcs in the case of one arbitrary fault from the faults scenario; constraints (3) and (8) ensure that the flow conservation conditions are met; constraints (4) and (9) are responsible for the non-negativity of flows, and constraints (5) and (10) restrict the choice of the capacities of arcs.

The optimal values of the capacities of arcs $Y^* = \{y_a^*, a \in A\}$ are determined by convex smooth or non-smooth functions $f_A(Y)$ and $f_B(Y)$. For example, linear function

$$F_A^1(Y) = \sum_{a \in A} c_a y_a,$$

where c_a is the cost of the unit capacity of arc a , can be used to find the capacities of arcs that minimize the total cost of their construction.

Quadratic function

$$F_A^2(Y) = \sum_{a \in A} (y_a - y_a^e)^2,$$

where $y_a^e, a \in A$ are the "desired" capacities of arcs, allows us to find the capacities of the arcs with a minimum deviation from $y_a^e, a \in A$ using the least squares method. The queuing theory uses smooth non-linear function

$$F_A^3(Y) = \sum_{a \in A} \frac{y_a}{u_a - y_a}$$

where u_a is the "nominal" capacity of the arc a . The value of the maximum flow in the network through the vertices V corresponds to non-smooth function

$$F_A^4(Y) = \max_{i \in V} \left(\sum_{a \in A_i^+} y_a + \sum_{a \in A_i^-} y_a \right).$$

Problems A and B are convex optimization problems of high dimension, even for small networks. For example, if $|V| \approx 40$, $|A| \approx 50$, $|T| \approx 50$, $|K| \approx 1000$ and $|P| \approx 2000$, then $N_A \approx 2\,500\,000$, $M_A \approx 2\,000\,000$ and $N_B \approx 100\,000$, $M_B \approx 50\,000$, where N_A and N_B are the number of variables, and M_A and M_B are the number of constraints in problems A and B, respectively. The block structure of both problems makes it possible to use for their solution decomposition methods in combination with modifications of r -algorithms in the same way as was done for LP-problems [2].

Using problems A and B one can also describe some situations that allow increasing the network arc capacities by μ times. Such situations are hardly can be called "emergency". From the standpoint of fault-tolerant networks, we omit such situations. However, in problems A and B they are completely valid and mean the following. At some point, it will be necessary to make the best decision about capacities of the network, taking into account the perspective of increasing capacities of those or other network arcs.

Computational Experiments for Linear Problems A and B

We illustrate the capabilities of models A and B using the example of two faults scenarios in the network $Net(6,8)$ for a linear objective function and one correspondence $d_{1,6} = 3$. The calculation results for model A are presented in Fig. 3–Fig. 6. The costs of transmission of units of flow in the network $Net(6,8)$ are shown in Fig. 3. Figure 4 shows the optimal capacity values of the arcs for a non-failure scenario. For this case, the path $P_1 = (1,3,5,6)$ for transmission of three units of flow is the shortest path in the network $Net(6,8)$ and goes along arcs (1,3), (3,5) and (5,6). The cost of transmission of a unit of flow along the shortest path is equal to eleven ($3 + 5 + 3 = 11$), therefore, to send three units of flow you need thirty three units of cost ($33 = 3 \times 11$). The remaining five arcs are not involved and do not play any role in the cost. But the situation changes if the flow transmission in the network is carried out under faults scenarios 0.5F and 1F (see Fig. 5–Fig. 6).

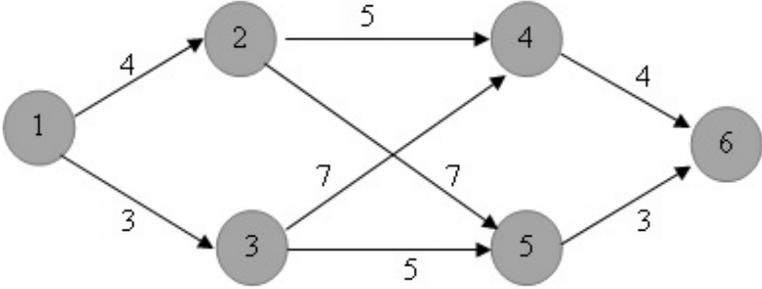


Fig 3. The costs of transmission of units of flow in the network $Net(6,8)$

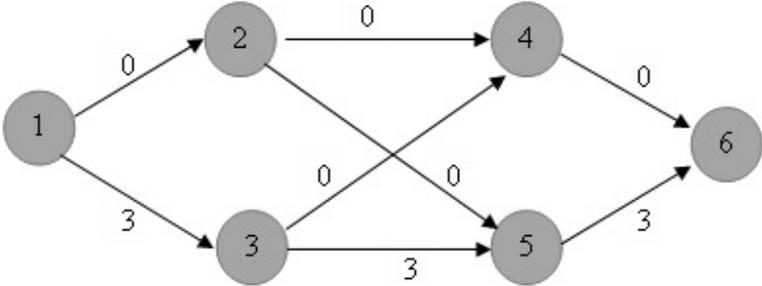


Fig 4. The optimal capacity values of arcs for non-failure scenario, $f_A^* = 33$

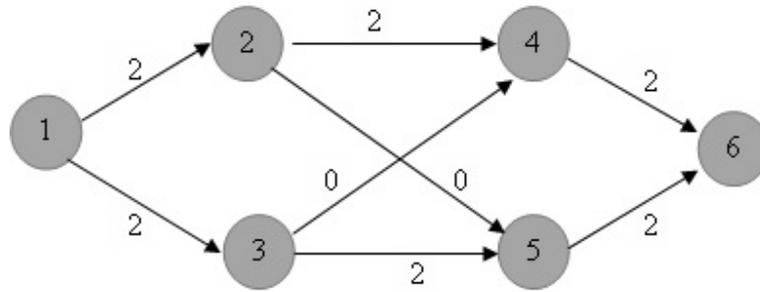


Fig 5. The scenario $0.5F$, $f_A^* = 48$

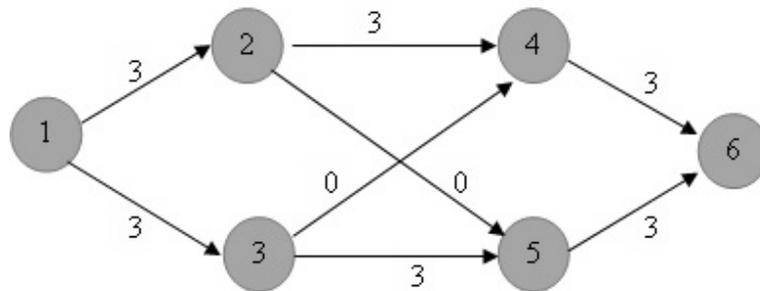


Fig 6. The scenario $1F$, $f_A^* = 72$

Figure 5 shows the solution to problem A with the fault scenario $0.5F$, that is, if one arbitrary arc in the network can reduce its capacity by half. Intuitively, in such a scenario, the value of objective function cannot be greater than sixty six ($33 \times 2 = 66$), that is, it will be no greater than a double optimal value of the problem without faults, where a flow of three units can be transmitted if to reduce the capacity of the shortest path from vertex 1 to vertex 6 in twice. In the optimal solution compared to this "intuitive" solution, eighteen units of cost are saved ($66 - 48 = 18$).

For transmitting the flow from vertex 1 to vertex 6, two paths are used: $P_1 = (1,3,5,6)$ and $P_4 = (1,2,4,6)$, that do not intersect along arcs. The path P_1 is the best, and the cost of transmitting a unit of flow is equal to eleven units. The path P_4 is more expensive, and its cost is thirteen units. In the first case (one from the three faults occurred either of the arc $(1,3)$, or the arc $(3,5)$, or the arc $(5,6)$), a unit of flow is sent along the path P_1 , and two units remaining forwarded along the path P_4 . The cost of sending

these two flows is equal to thirty seven units ($11 + 2 \times 13 = 37$). In the second case (one from the three faults occurred either of the arc (1,2), or the arc (2,4), or the arc (4,6)), two units of flow are sent along the P_1 path, and the rest unit of flow is sent along the path P_4 . The cost of sending both flows is equal to thirty five units ($2 \times 11 + 13 = 35$) and will be less than in the first case. So, for the fault scenario 0.5F, the first case determines the reserves of the arc capacity so that the network $Net(6,8)$ would be fault tolerant.

A similar situation is observed for the scenario 1F, if only one arbitrary arc fails (Fig. 6). In this case, for sending a flow in three units, the same two paths are used. At the same time, the cost of transmitting the flow grows more than twice, because the capacity of the arcs is reserved so that the total volume of the flow can be transmitted either by the shortest path P_1 or by more expensive path P_4 .

In both faults scenarios, arcs (2,4) and (3,5) are not used to transfer the flow from vertex 1 to vertex 6. They are redundant and do not affect the cost of the network, so the cost of sending a unit of flow along the paths $P_2 = (1,3,4,6)$ and $P_3 = (1,2,5,6)$ equals to fourteen units and is greater than for the P_1 and P_4 paths. Therefore, the paths P_1 and P_4 determine the optimal logical network structure of the network $Net(6,8)$ for scenarios 0.5F and 1F, no matter how much flow you need to send from vertex 1 to vertex 6.

This observation is confirmed by the values of f_P^* , which are presented in Table 2 in the first columns for the scenarios 0.5F and 1F. They are associated with those solutions of the problem B, where only the paths P_1 and P_4 are selected to transmit the flow in the network. As we see, the values of f_A^* and f_P^* are the same, which proves the optimality of the logical structure $P_1 \& P_4$. The relative difference of f_P^* and f_A^* is the value characterizing the degree of "non-optimality" of the logical structure of the network. So, for the scenario 1F, the degree of "non-optimality"

$$\omega_{AB} = \frac{f_P^* - f_A^*}{f_A^*} = \frac{99 - 72}{72} = 0,375 = 37.5\%$$

is the same as for P_1, P_2, P_3 and P_2, P_3, P_4 . For the scenario 0.5F, the "non-

optimality" levels will be different:

$$\omega_{AB}^{123} = \frac{62 - 48}{48} \approx 0,29167 \approx 29\%;$$

$$\omega_{AB}^{234} = \frac{56 - 48}{48} \approx 0,178 \approx 18\%.$$

TABLE II. PROBLEM B FOR SCENARIOS 0,5F AND 1F

arcs		Paths for scenario 0,5F			Paths for scenario 1F		
	(i, j)	(1,4)	(1,2,3)	(2,3,4)	(1,4)	(1,2,3)	(2,3,4)
1	(1,2)	2,0	2	2	3,0	3,0	3.0
2	(1,3)	2,0	2	2	3,0	3,0	3.0
3	(2,4)	2,0	2	0	3,0	3,0	0.0
4	(2,5)	0,0	0	2	0,0	3,0	3.0
5	(3,4)	0,0	2	0	0,0	3,0	3.0
6	(3,5)	2,0	0	2	3,0	0,0	3.0
7	(4,6)	2,0	6	0	3,0	3,0	3.0
8	(5,6)	2,0	0	6	3,0	3,0	3.0
$f_B^* =$		48,00	62	56	72,00	99,00	99,00

Conclusions

Mathematical models, methods and software for two classes of problems for finding the capacities of arcs of a fault-tolerant oriented network are described. In the first class of problems (**problem A**), all possible paths in the network are involved in transferring flows. In the second class of problems (**problem B**), only paths from a predetermined set of paths are involved in transferring flows. Mathematical models of the problems are represented by nonlinear convex optimization problems with the network structure of the constraint matrix. The developed algorithms and software are based on implementations of the subgradient method with space transformation (r-algorithm) and take into account structural features of optimization models. They can be used to make decisions on the choice of parameters in design and planning of future development of fault-

tolerant transport, telecommunication, and other networks.

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References

[1] Shor N.Z. Nondifferentiable optimization and polynomial problems. Boston; Dordrecht; London: Kluwer Academic Publishers. 1998. 412 p.

[2] The problems of optimal design of reliable networks / N.Z. Shor, I.V. Sergienko, V.P. Shylo, P.I. Stetsyuk and others. – K.: Nauk. dumka, 2005. – 230 p. (in Ukrainian)

[3] Stetsyuk P. I. Theory and Software Implementations of Shor's r-Algorithms, *Cybern. Syst. Analysis*, Vol. 53, No. 5, 692–703 (2017).

[4] Stetsyuk P. I. Shor's r-Algorithms: Theory and Practice. In: *Optimization Methods and Applications*: Ed. by S. Butenko, P.M. Pardalos, V. Shylo. Springer. 2017. P. 495–520.

[5] Sergienko Ivan V. *Methods of optimization and systems analysis for problems of transcomputational complexity*. – New York, Heidelberg, Dordrecht, London: Springer, 2012. – 226p.