V.M. GLUSHKOV INSTITUTE
OF CYBERNETICS OF THE NAS OF UKRAINE

SIMPLEX METHOD: PART 2

LECTURE 7/SURVEY OF OPTIMIZATION

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AGENDA

- Major Methods of Solving LP
- Recognizing optimality , Recognizing optimality principle
- Recognizing unbounded LPs, Degeneracy and cycling
- Properties of LP dictionaries and the simplex method
- Heavenly Pouch Inc. Problem Modified: Simplex Method
- Big-M and Two-Phase Simplex Methods

MAJOR METHODS OF SOLVING LP

- The Graphical Method (for 2 variables)
- The Substitution Method
- The Linear Combination Method
- Simplex Method
- Khachiyan's Algorithm (polynomial-time solvability of linear programs)
- The ellipsoid methods (N. Shor's methods)
- Karmarkar's algorithm (an interior-point method)

HEAVENLY POUCH INC. PROBLEM: SIMPLEX METHOD

Step 2 Dictionary:

$$z = 9{,}375 - \frac{25}{4}s_2 - \frac{15}{4}s_3$$

$$s_1 = 25 - \frac{1}{4}s_2 + \frac{1}{4}s_3$$

$$x_2 = 300 - s_2$$

$$x_1 = 125 + \frac{5}{4}s_2 - \frac{1}{4}s_3$$

$$s_4 = 225 - \frac{5}{4}s_2 + \frac{1}{4}s_3$$

Step 2 Tableau:

z	x_1	x_2	s_1	s_2	s_3	s_4	rhs	Basis	Ratio
1	0	0	0	25/4	15/4	0	9,375	z	
0	0	0	1	1/4	-1/4	0	25	s_1	
0	0	1	0	1	0	0	300	x_2	
0	1	0	0	-5/4	1/4	0	125	x_1	
0	0	0	0	5/4	-1/4	1	225	s_4	

HEAVENLY POUCH INC. PROBLEM: SIMPLEX METHOD

Step 2 basic feasible solution	
BV_2 :	x_1, x_2, s_1 , s_4
NV_2 :	s_3, s_2
b f s :	$x_1 = 125; x_2 = 300$ $s_1 = 25, s_2 = 0, s_3 = 0, s_4 = 225$ $z = 9,375$

RECOGNIZING OPTIMALITY

Step 2 Dictionary:

$$z = 9{,}375 - \frac{25}{4}s_2 - \frac{15}{4}s_3$$

,where both s_2 and s_3 are nonnegative

$$s_1 = 25 - \frac{1}{4}s_2 + \frac{1}{4}s_3$$

$$x_2 = 300 - s_2$$

$$x_1 = 125 + \frac{5}{4}s_2 - \frac{1}{4}s_3$$

$$s_4 = 225 - \frac{5}{4}s_2 + \frac{1}{4}s_3$$

The highest possible z is obtained when:

$s_2 = s_3 = 0$

The slack variables are ignored since they are not a part of the original LP.

RECOGNIZING OPTIMALITY PRINCIPLE

If in a feasible dictionary, all nonbasic variables have *nonpositive* coefficients in the z-row, then the corresponding basic feasible solution is an optimal solution of the LP.

If we use the tableau format, then the basic feasible solution is optimal if all non-basic variables have *nonnegative* coefficients in row 0 of the corresponding tableau.

RECOGNIZING UNBOUNDED LPs EXAMPLE 1:

Dictionary Format:

$$z = 90 - 25x_1 + 4x_2$$

The only entering variable candidate is x_2.

$$s_1 = 25 - 14x_1 + x_2$$
$$s_2 = 30 - x_1$$
$$s_3 = 12 + 5x_1 + 14x_2$$
$$s_4 = 22 - 4x_1 + 7x_2$$

positive ⇒ x_2, z increases to $+\infty$

⇓

the problem is unbounded

RECOGNIZING UNBOUNDED LPs EXAMPLE 1:

Tableau Format:

z	x_1	x_2	s_1	s_2	s_3	s_4	rhs	Basis
1	25	- 4	0	0	0	0	90	z
0	14	- 1	1	0	0	0	25	s_1
0	1	0	0	1	0	0	30	s_2
0	-5	- 14	0	0	1	0	12	s_3
0	4	- 7	0	0	0	1	22	s_4

RECOGNIZING UNBOUNDED LPs PRINCIPLE

If during the execution of the simplex method we encounter a variable that has all nonnegative coefficients in the *dictionary* format, then the LP is unbounded.

In *tableau* format, an LP is proved to be unbounded as soon as a column with no positive entries is detected.

DEGENERACY AND CYCLING Example 2:

maximize $5x_1 + 4x_2 - 20x_3 - 2x_4$

subject to $\frac{1}{4}x_1 - \frac{1}{8}x_2 + 12x_3 + 10x_4 \leq 0$

$$\frac{1}{10}x_1 + \frac{1}{20}x_2 + \frac{1}{20}x_3 + \frac{1}{5}x_4 \leq 0$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Let x_5, x_6 – slack variables

DEGENERACY AND CYCLING Example 2:

Step 0 Tableau format:

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs	Basis
1	- 5	- 4	20	2	0	0	0	z
0	$\frac{1}{4}$	$-\frac{1}{8}$	12	10	1	0	0	x_5
0	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{5}$	0	1	0	x_6

Both basic variables are equal to 0.

The entering variable

The leaving variable

DEGENERACY AND CYCLING EXAMPLE 2:

Definition: Basic solutions with one or more basic variables equal to 0 are called *degenerate*.

Step 1 Tableau format:

The basic feasible solution is the same as at step 0, even though the basis has changed

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs	Basis
1	0	$-\frac{13}{2}$	260	202	20	0	0	z
0	1	$-\frac{1}{2}$	48	46	4	0	0	x_1
0	0	$\frac{1}{10}$	$-\frac{19}{4}$	$-\frac{19}{5}$	$-\frac{2}{5}$	1	0	x_6

DEGENERACY AND CYCLING EXAMPLE 2:

Definition: An iteration of the simplex method, which results in a new basis with the basic feasible solution that is identical to the previous basic feasible solution is called *a degenerate iteration* and the corresponding phenomenon is referred to *as degeneracy.*

Step 2 Tableau format:

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs	Basis
1	0	0	$-\frac{195}{4}$	-45	- 6	65	0	z
0	1	1	$\frac{97}{4}$	21	2	5	0	x_1
0	0	0	$-\frac{95}{2}$	- 38	- 4	10	0	x_2

DEGENERACY AND CYCLING EXAMPLE 2:

Step 3 Tableau format:

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs	Basis
1	$\frac{195}{97}$	0	0	$-\frac{270}{97}$	$-\frac{190}{97}$	$\frac{7280}{97}$	0	z
0	$\frac{190}{97}$	1	0	$\boxed{\frac{304}{97}}$	$\frac{8}{97}$	$\frac{1920}{97}$	0	x_1
0	$\frac{4}{97}$	0	1	$\frac{84}{97}$	$\frac{8}{97}$	$\frac{20}{97}$	0	x_2

DEGENERACY AND CYCLING EXAMPLE 2:

Step 4 Tableau format:

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs	Basis
1	$\frac{15}{4}$	$\frac{135}{152}$	0	0	$-\frac{39}{19}$	$\frac{1760}{19}$	0	z
0	$-\frac{1}{2}$	$-\frac{21}{76}$	1	0	$\frac{2}{19}$	$-\frac{100}{19}$	0	x_3
0	$\frac{5}{8}$	$\frac{97}{304}$	0	1	$-\frac{1}{38}$	$\frac{120}{19}$	0	x_4

DEGENERACY AND CYCLING EXAMPLE 2:

Step 5 Tableau format:

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs	Basis
1	-6	$-\frac{9}{2}$	$\frac{39}{2}$	0	0	-10	0	z
0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	1	0	5	0	x_4
0	$-\frac{19}{4}$	$-\frac{21}{8}$	$\frac{19}{2}$	0	1	-50	0	x_5

DEGENERACY AND CYCLING EXAMPLE 2:

Step 6 Tableau format:

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs	Basis
1	- 5	- 4	20	2	0	0	0	z
0	$\frac{1}{4}$	$-\frac{1}{8}$	12	10	1	0	0	x_5
0	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{5}$	0	1	0	x_6

Exactly the same as the step 0 tableau

DEGENERACY AND CYCLING EXAMPLE 2:

Thus, if we continue with the execution of the simplex method, we will keep repeating the calculations performed in steps 1–6 and will never be able to leave the same solution.

Definition. A situation when the simplex method goes through a series of degenerate steps, and as a result, revisits a basis it encountered previously is called cycling.

METHODS OF AVOIDING CYCLING: BLAND'S RULE

- Bland's rule

1) Select the entering variable by choosing the smallest index nonbasic variable whose simplex direction is improving.
2) Select the leaving variable as the smallest index from those that define the maximal step size. (x, s)=(0, b).

METHODS OF AVOIDING CYCLING: BLAND's RULE

- **Theorem:** If Bland's rule is used to select the entering and leaving variables in the simplex method, then cycling never occurs.

Step 5 Tableau format:

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs	Basis
1	-6	$-\frac{9}{2}$	$\frac{39}{2}$	0	0	-10	0	z
0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	1	0	5	0	x_4
0	$-\frac{19}{4}$	$-\frac{21}{8}$	$\frac{19}{2}$	0	1	-50	0	x_5

Candidates for entering the basis: x_1, x_2, x_6

METHODS OF AVOIDING CYCLING: BLAND'S RULE

Step 6 Tableau format:

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs	**Basis**
1	0	$-\frac{3}{2}$	$\frac{45}{2}$	12	0	50	0	z
0	1	$\frac{1}{2}$	$\frac{1}{2}$	2	0	10	0	x_1
0	0	$-\frac{1}{4}$	$\frac{95}{8}$	$\frac{19}{2}$	1	$-\frac{5}{2}$	0	x_5

METHODS OF AVOIDING CYCLING: BLAND'S RULE

Optimal Tableau format:

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs	Basis
1	3	0	24	18	0	80	0	z
0	2	1	1	4	0	20	0	x_2
0	$\frac{1}{2}$	0	$\frac{97}{8}$	$\frac{21}{2}$	1	$\frac{5}{2}$	0	x_5

POTENTIAL ISSUES

- INITIALIZATION: Will we always be able to start? How to find the starting feasible dictionary?
- ITERATION: Can we always choose an entering variable, find the leaving variable and construct the next feasible dictionary by pivoting?
- TERMINATION: Is there a possibility that the simplex method will construct an endless sequence of solutions without ever reaching an optimal solution?

PROPERTIES OF LP DICTIONARIES AND THE SIMPLEX METHOD

Property 1: Any two dictionaries of the same LP with the same basis are identical.

Any two dictionaries with the same basis must be identical.
Consider two dictionaries corresponding to the same basis. Let B be the set of indices of the basic variables and let N be the set of indices of nonbasic variables.

$$z = \bar{z} + \sum_{j \in N} \overline{c_j}\, x_j$$

$$x_i = \overline{b_i} - \sum_{j \in N} \overline{a_{ij}}\, x_j \quad , i \in B$$

$$z = \tilde{z} + \sum_{j \in N} \widetilde{c_j}\, x_j$$

$$x_i = \widetilde{b_i} - \sum_{j \in N} \widetilde{a_{ij}}\, x_j \quad , i \in B$$

PROPERTIES OF LP DICTIONARIES AND THE SIMPLEX METHOD

For a nonbasic variable x_k , set $x_k = t > 0$, and for $j \in N$, $j \neq k$, set $x_j = 0$.

$$\bar{b}_i - \bar{a}_{ik}t = \tilde{b}_i - \tilde{a}_{ik}t \quad \forall i \in \boldsymbol{B}, \quad \boldsymbol{and} \ \ \bar{z} + \bar{c}_k t = \tilde{z} + \tilde{c}_k t$$

Since these identities hold for any t, we have

$$\bar{b}_i = \tilde{b}_{i,} \ \bar{a}_{ik} = \tilde{a}_{ik}, \ \bar{z} = \tilde{z}, \bar{c}_k = \tilde{c}_k, \ \forall i \in B, k \in N$$

■

PROPERTIES OF LP DICTIONARIES AND THE SIMPLEX METHOD

Property 2: Every solution of the set of equations comprising the dictionary obtained at any step of the simplex method is also a solution of the step 0 dictionary, and vice versa.

■

The initial dictionary is a linear system that represents the original LP written in the standard form. The only transformations we apply to this linear system at each subsequent iteration of the simplex method are the elementary row operations used to express the new set of basic variables through the remaining variables. Since applying an elementary row operation to a linear system results in an equivalent linear system, the property is true.

■

PROPERTIES OF LP DICTIONARIES AND THE SIMPLEX METHOD

Property 3: If step 0 dictionary is feasible, then each consecutive dictionary generated using the simplex method is feasible.

The ratio test is used to determine the leaving variable at each step and is designed to ensure that the constant term in the right-hand side of each equation is nonnegative, so that setting all the nonbasic variables to 0 yields nonnegative values for all the basic variables and thus the corresponding basic solution is feasible.
Thus, if we start with a feasible dictionary, feasibility is preserved throughout execution of the simplex method.

PROPERTIES OF LP DICTIONARIES AND THE SIMPLEX METHOD

The simplex method may go through some consecutive degenerate iterations with no change in the objective function value, in which case cycling can occur. It appears that this is the only case where the method may not terminate.

Theorem. If the simplex method avoids cycling, it must terminate by either finding an optimal solution or by detecting that the LP is unbounded.

There are only C_{n+m}^{m} different ways of choosing a set of m basic variables from the set of n+m variables. Since any two dictionaries corresponding to the same basis are identical, there can only be a finite number of the simplex method steps that are different.
Therefore, if the simplex method does not terminate, it must eventually revisit a previously visited basis, meaning that cycling occurs.

HEAVENLY POUCH INC. PROBLEM MODIFIED

Additional constraint: at least 100 carriers are made.

Maximize	$15x_1 + 25x_2$	(profit)
Subject to (s.t.)	$x_1 + x_2 \leq 450$	(solid color fabric constraint)
	$x_2 \leq 300$	(printed fabric constraint)
	$4x_1 + 5x_2 \leq 2000$	(budget constraint)
	$x_1 + x_2 \geq 100$	(manufacturing constraint)
	$x_1 \leq 350$	(demand constraint)
	$x_1, x_2 \geq 0$	(nonnegativity constraints)

HEAVENLY POUCH INC. PROBLEM: SIMPLEX METHOD

Convert the LP to the standard form by introducing a slack variable s_i for each constraint i, i = 1,...,4 and an excess variable e_5 for the fifth constraint:

Maximize $15x_1 + 25x_2$

Subject to (s.t.)

$$
\begin{aligned}
x_1 + x_2 + s_1 &= 450 \\
x_2 + s_2 &= 300 \\
4x_1 + 5x_2 + s_3 &= 2000 \\
x_1 + s_4 &= 350 \\
x_1 + x_2 - e_5 &= 100 \\
x_1, x_2, s_1, s_2, s_3, s_4, e_5 &\geq 0
\end{aligned}
$$

HEAVENLY POUCH INC. PROBLEM: SIMPLEX METHOD

Dictionary format modified

$$z = 15x_1 + 25x_2$$

$$s_1 = 450 - x_1 - x_2$$
$$s_2 = 300 - x_2$$
$$s_3 = 2{,}000 - 4x_1 - 5x_2$$
$$s_4 = 350 - x_1$$
$$e_5 = -100 + x_1 + x_2$$

$x_1 = x_2 = 0$ → $e_5 = -100 < 0$

→ Infeasible

we cannot initialize the simplex method with this dictionary.

THE SIMPLEX METHOD FOR A GENERAL LP

- the big-M method
- the two-phase simplex method

introduce an artificial variable for each constraint where the starting basic variable (such as the slack or excess variable) is not readily available

THE SIMPLEX METHOD FOR A GENERAL LP

Include an artificial variable $a_5 \geq 0$ as follows:

$$x_1 + x_2 - e_5 = 100 \longrightarrow x_1 + x_2 - e_5 + a_5 = 100$$

Make sure that in the end the artificial variable $a_5 = 0$.

The both methods utilize alternative ways of driving the artificial variables out of the basis, thus ensuring that they all eventually vanish whenever the LP is feasible.

THE SIMPLEX METHOD FOR A GENERAL LP

Consider a general LP:

Maximize $\sum_{j=1}^{n} c_j x_j$

subject to $\sum_{j=1}^{n} a_{ij}^{`} x_j \le b_i^{`}$, i=1,…, $m^{`}$

$\sum_{j=1}^{n} a_{ij}^{``} x_j \le b_i^{``}$, i=1,…, $m^{``}$

Introduce $m^{`}$ slack variables

Maximize $\sum_{j=1}^{n+m^{`}} c_j x_j$

subject to $\sum_{j=1}^{n+m^{`}} a_{ij}^{`} x_j \le b_i^{`}$, i=1,…, $m^{`}$

$x_j \ge 0, \quad j = 1, \ldots, n + m^{`}$

where m = m`+ m `` , indices i = m`+ 1,...,m correspond to the original equality constraints, and $b_i \ge 0$ ∀i = m`+ 1,...,m

BIG-M METHOD ALGORITHM

1) Modify the constraints so that the RHS of each constraint is nonnegative (This requires that each constraint with a negative RHS be multiplied by -1. Remember that if you multiply an inequality by any negative number, the direction of the inequality is reversed!). After modification, identify each constraint as ≥ , ≤ or = constraint.

2) Convert each inequality constraint to standard form. If constraint i is ≤ constraint, we add a slack variable s_i ; and if constraint i is ≥ constraint, we subtract an excess (surplus) variable e_i .

BIG-M METHOD ALGORITHM

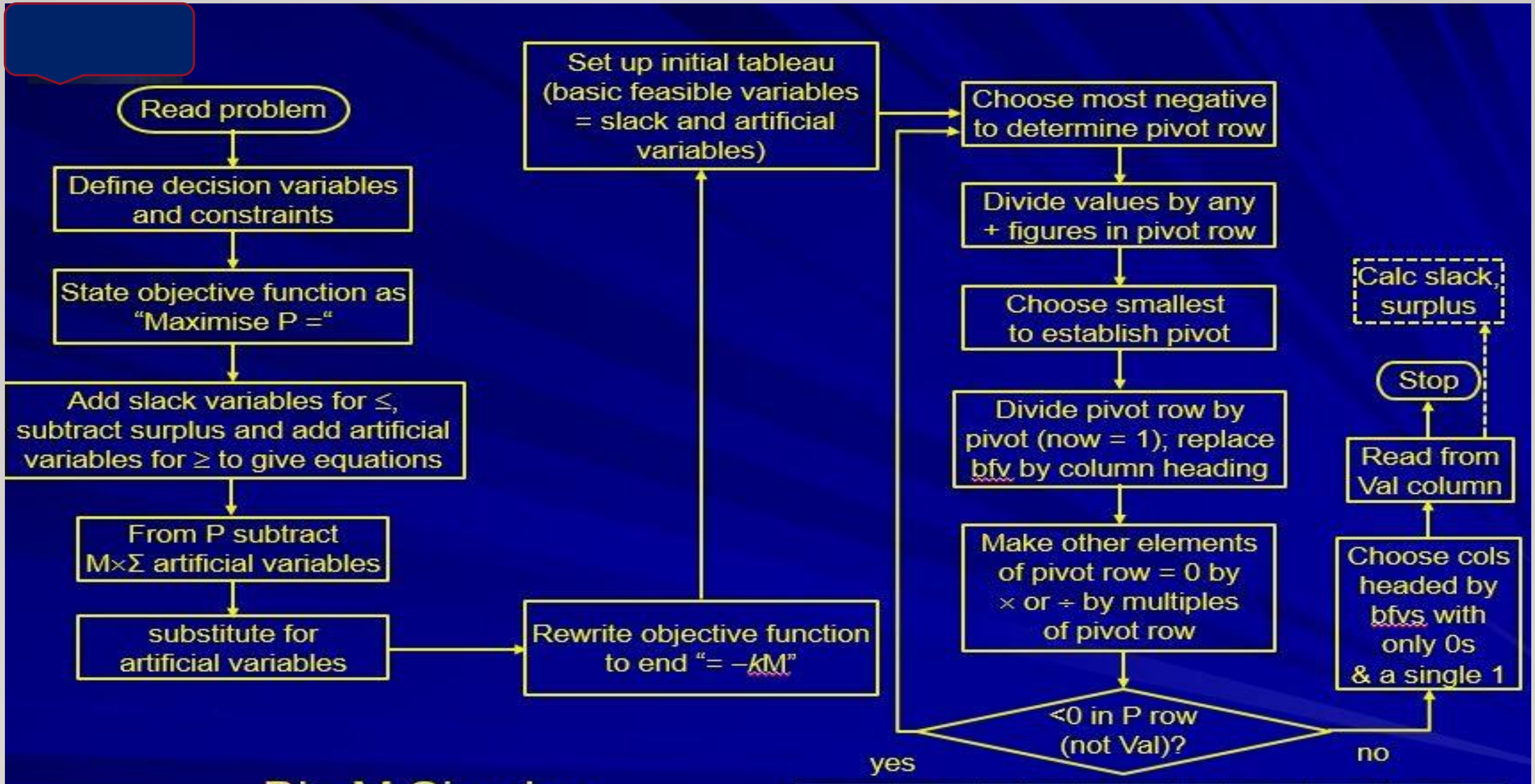
3) Add an artificial variable a_i to the constraints identified as ≥ or = constraints at the end of Step 1. Also add the sign restriction $a_i \geq 0$.

4) Let M denote a very large positive number. If the LP is a min problem, add (for each artificial variable) Ma_i to the objective function (before equal to 0). If the LP is a max problem, add (for each artificial variable) $-Ma_i$ to the objective function (before equal to 0).

BIG-M METHOD ALGORITHM

5) Since each artificial variable will be in the starting basis, all artificial variables must be eliminated from objective row before beginning the simplex. Now solve the transformed problem by the simplex (In choosing the entering variable, remember that M is a very large positive number!).

If all artificial variables are equal to zero in the optimal solution, we have found the optimal solution to the original problem. If any artificial variables are positive in the optimal solution, the original problem is infeasible.



Basic variables	x	y	z	s_1	s_2	s_3	a_1	value	(working)
z	0	1	1	1	0	2	-2	12	
s_2	0	-1	0	1	1	3	-3	15	
x	1	0	0	0	0	-1	1	4	
P	0	2	0	1	0	1	M - 1	16	

BIG M METHOD *EXAMPLE*

Maximize $x_1 - 2x_2 + 3x_3$

subject to
$$-2x_1 + 3x_2 + 4x_3 \geq 12$$
$$3x_1 + 2x_2 + x_3 \geq 6$$
$$x_1 + x_2 + x_3 \leq 9$$
$$x_1, x_2, x_3 \geq 0$$

Convert to Standard form

Maximize $x_1 - 2x_2 + 3x_3$

subject to
$$-2x_1 + 3x_2 + 4x_3 - x_4 = 12$$
$$3x_1 + 2x_2 + x_3 - x_5 = 6$$
$$x_1 + x_2 + x_3 + x_6 = 9$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

the basic solution with the basis consisting of x_4, x_5, and x_6 is infeasible

BIG M METHOD *EXAMPLE*

Introduce *artificial* variables for the first two constraints a_1, and a_2

the corresponding auxiliary problem can be written as a maximization problem

$$\text{Maximize} \quad -a_1 - a_2$$

$$\begin{aligned} \text{subject to} \quad -2x_1 + 3x_2 + 4x_3 - x_4 \quad\quad & + a_1 = 12 \\ 3x_1 + 2x_2 + x_3 \quad - x_5 \quad\quad & + a_2 = 6 \\ x_1 + x_2 + x_3 \quad\quad + x_6 \quad & = 9 \end{aligned}$$

$$x_1, x_2, x_3, x_4, x_5, x_6, a_1, a_2 \geq 0$$

the objective function of the auxiliary problem is always nonnegative; thus, any feasible solution of this problem with $a_i = 0$ is feasible.

BIG *M* METHOD *EXAMPLE*

The big-M problem associated with LP is given by:

$$\text{Maximize} \quad x_1 - 2x_2 + 3x_3 \qquad - M a_1 - M a_2$$

$$\text{subject to} \quad -2x_1 + 3x_2 + 4x_3 - x_4 \qquad + a_1 \qquad = 12$$

$$3x_1 + 2x_2 + x_3 \qquad -x_5 \qquad + a_2 = 6$$

$$x_1 + x_2 + x_3 \qquad +x_6 \qquad = 9$$

$$x_1, x_2, x_3, x_4, x_5, x_6, a_1, a_2 \geq 0$$

If a feasible solution for the big-M with $a_i > 0$ for at least one $i \in \mathcal{J}^a$ exists, the objective function value can be made arbitrarily poor (i.e., very large, negative) by selecting a sufficiently large constant $M > 0$.

BIG *M* METHOD *EXAMPLE*

Step 0 Tableau format:

M- nonnumerical parameter: $M \longrightarrow \infty$
While comparing two nonbasic variable coefficients involving M, the coefficient with a greater multiplier for M is considered greater

z	x_1	x_2	x_3	x_4	x_5	x_6	a_1	a_2	rhs	Basis
1	- M-1	5M+2	-5M-3	M	M	0	0	0	-18M	z
0	−2	3	4	-1	0	0	1	0	12	a_1
0	3	2	1	0	-1	0	0	1	6	a_2
0	1	1	1	*0*	0	1	0	0	9	x_6

BIG *M* METHOD *EXAMPLE*

Step 1 Tableau format:

z	x_1	x_2	x_3	x_4	x_5	x_6	a_1	a_2	rhs	Basis
1	$-\frac{7M+5}{2}$	$-\frac{5M-17}{4}$	0	$-\frac{M+3}{4}$	M	0	$\frac{5M+3}{4}$	0	-3M+9	z
0	$-\frac{1}{2}$	$\frac{3}{4}$	1	$-\frac{1}{4}$	0	0	$\frac{1}{4}$	0	12	a_1
0	$\frac{7}{2}$	$\frac{5}{4}$	0	$\frac{1}{4}$	- 1	0	$-\frac{1}{4}$	1	3	a_2
0	$\frac{3}{2}$	$\frac{1}{4}$	0	$\frac{1}{4}$	0	1	$-\frac{1}{4}$	0	6	x_6

BIG *M* METHOD *EXAMPLE*

Step 2 Tableau format:

z	x_1	x_2	x_3	x_4	x_5	x_6	a_1	a_2	rhs	Basis
1	0	$\frac{36}{7}$	0	$-\frac{4}{7}$	$-\frac{5}{7}$	0	$\frac{7M+4}{7}$	$\frac{7M+5}{7}$	$\frac{78}{7}$	z
0	0	$\frac{13}{14}$	1	$-\frac{13}{14}$	$-\frac{1}{7}$	0	$\frac{3}{14}$	$\frac{1}{7}$	$\frac{24}{7}$	x_3
0	1	$\frac{5}{14}$	0	$\frac{1}{14}$	$-\frac{2}{7}$	0	$-\frac{1}{14}$	$\frac{2}{7}$	$\frac{6}{7}$	x_1
0	0	$-\frac{2}{7}$	0	$\frac{1}{7}$	$\frac{3}{7}$	1	$-\frac{1}{7}$	$-\frac{3}{7}$	$\frac{33}{7}$	x_6

BIG *M* METHOD *EXAMPLE*

Step 3 Tableau format:

z	x_1	x_2	x_3	x_4	x_5	x_6	a_1	a_2	rhs	Basis
1	0	$\frac{14}{3}$	0	$-\frac{1}{3}$	0	$\frac{5}{3}$	$\frac{3M+1}{3}$	M	19	z
0	0	$\frac{5}{6}$	1	$-\frac{1}{6}$	0	$\frac{1}{3}$	$\frac{1}{6}$	0	5	x_3
0	1	$\frac{1}{6}$	0	$\frac{1}{6}$	0	$\frac{2}{3}$	$-\frac{1}{6}$	0	4	a_2
0	$\frac{3}{2}$	$\frac{1}{4}$	0	$\frac{1}{4}$	0	1	$-\frac{1}{4}$	0	6	x_6

BIG M METHOD *EXAMPLE*

Step 4 Tableau format - optimal:

z	x_1	x_2	x_3	x_4	x_5	x_6	a_1	a_2	rhs	Basis
1	2	5	0	0	0	3	M	M	27	z
0	1	1	1	0	0	1	0	0	9	x_3
0	6	1	0	1	0	4	- 1	0	24	x_4
0	-2	-1	0	0	1	1	0	-1	3	x_5

The optimal solution : $x_1^* = x_2^* = 0;\ x_3^* = 9;\ z^* = 27$

THE TWO-PHASE SIMPLEX METHOD ALGORITHM

1) Modify the constraints so that the right-hand side of each constraint is nonnegative. This requires that each constraint with a negative right-hand side be multiplied through by -1.

2) Identify each constraint that is now (after step 1) ≥ or = constraint. In step 3, an artificial variable will be added to each constraint.

3) Convert each inequality constraint to the standard form. If constraint i is ≤ constraint, then add a slack variable s_i . If constraint i is ≥ constraint, subtract an excess variable e_i.

THE TWO-PHASE SIMPLEX METHOD ALGORITHM

4) If (after step 2) constraint i is ≥ or = constraint, add an artificial variable a_i .

5) For now, ignore the original LP's objective function. Instead solve an LP whose objective function is min W = (sum of all the artificial variables). This is called *the Phase I LP*. The act of solving the Phase I LP will force the artificial variables to be zero.

THE TWO-PHASE SIMPLEX METHOD ALGORITHM

Because each $a_i \geq 0$, solving the Phase I LP will result in one of the following three cases:

Case 1 The optimal value of W is greater than zero. In this case, the original LP has no feasible solution.

Case 2 The optimal value of W is equal to zero, and no artificial variables are in the optimal Phase I basis. In this case, we drop all columns in the optimal Phase I tableau that corresponds to the artificial variables. We now combine the original objective function with the constraints from the optimal Phase I tableau. This yields the Phase II LP. The optimal solution to the Phase II LP is the optimal solution to the original LP.

THE TWO-PHASE SIMPLEX METHOD ALGORITHM

Case 3 The optimal value of W is equal to zero and at least one artificial variable is in the optimal Phase I basis. In this case, we can find the optimal solution to the original LP if at the end of Phase I we drop from the optimal Phase I tableau all non-basic artificial variables and any variable from the original problem that has a negative coefficient in objective row of the optimal Phase I tableau.

$W > 0$ corresponds to the original LP having no feasible solutions.

$W = 0$ corresponds to the original LP having at least one feasible solution.

THE TWO-PHASE SIMPLEX METHOD

The two-phase simplex method consists of the following two phases:

Phase I: Solve the *auxiliary problem* , and, as a result, either obtain a feasible tableau for the original problem (if the optimal objective value is 0), or conclude that the problem is infeasible (if the optimal objective value is positive).

Phase II: If the problem was not judged infeasible in Phase I, solve the original problem using the *optimal tableau of the auxiliary problem* to get the starting tableau for the original LP.

TWO-PHASE SIMPLEX METHOD *EXAMPLE*

Maximize $x_1 - 2x_2 + 3x_3$

subject to

$$-2x_1 + 3x_2 + 4x_3 \geq 12$$
$$3x_1 + 2x_2 + x_3 \geq 6$$
$$x_1 + x_2 + x_3 \leq 9$$
$$x_1, x_2, x_3 \geq 0$$

Convert to Standard form

Maximize $x_1 - 2x_2 + 3x_3$

subject to

$$-2x_1 + 3x_2 + 4x_3 - x_4 = 12$$
$$3x_1 + 2x_2 + x_3 - x_5 = 6$$
$$x_1 + x_2 + x_3 + x_6 = 9$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

the basic solution with the basis consisting of x_4, x_5, and x_6 is infeasible

THE TWO-PHASE SIMPLEX METHOD EXAMPLE

Phase I LP

Express the objective through non-basic variables:

$$
\begin{aligned}
z &= -a_1 - a_2 \\
&= -(12 + 2x_1 - 3x_2 - 4x_3 + x_4) - (6 - 3x_1 - 2x_2 - x_3 + x_5) \\
&= -18 + x_1 + 5x_2 + 5x_3 - x_4 - x_5,
\end{aligned}
$$

THE TWO-PHASE SIMPLEX METHOD EXAMPLE

Setting up the initial tableau of Phase I after eliminating artificial variables from z-row:

Step 0 Tableau (Phase I):

z	x_1	x_2	x_3	x_4	x_5	x_6	a_1	a_2	rhs	Basis
1	-1	-5	-5	1	1	0	0	0	-18	z
0	−2	3	4	-1	0	0	1	0	12	a_1
0	3	2	1	0	-1	0	0	1	6	a_2
0	1	1	1	0	0	1	0	0	9	x_6

THE TWO-PHASE SIMPLEX METHOD EXAMPLE

Step 1 Tableau (Phase I): (apply Simplex iterations)

z	x_1	x_2	x_3	x_4	x_5	x_6	a_1	a_2	rhs	Basis
1	$\frac{13}{2}$	0	$-\frac{5}{2}$	1	$-\frac{3}{2}$	0	0	$\frac{5}{2}$	-3	z
0	$-\frac{13}{2}$	0	$\frac{5}{2}$	-1	$-\frac{3}{2}$	0	1	$\frac{3}{2}$	3	a_1
0	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	3	x_2
0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	6	x_6

THE TWO-PHASE SIMPLEX METHOD EXAMPLE

Step 2 Tableau (Phase I) - optimal:

z	x_1	x_2		x_4	x_5	x_6	a_1	a_2	rhs	Basis
1	0	0	0	0	0	0	0	1	0	z
0	$-\frac{13}{5}$	0	1	$-\frac{2}{5}$	$\frac{3}{5}$	0	$\frac{2}{5}$	$-\frac{3}{5}$	$\frac{6}{5}$	x_3
0	$\frac{14}{5}$	1	0	$\frac{1}{5}$	$-\frac{4}{5}$	0	$-\frac{1}{5}$	$\frac{4}{5}$	$\frac{12}{5}$	x_2
0	$\frac{4}{5}$	0	0	$\frac{1}{5}$	$\frac{1}{5}$	1	$-\frac{1}{5}$	$-\frac{1}{5}$	$\frac{27}{5}$	x_6

THE TWO-PHASE SIMPLEX METHOD EXAMPLE

Drop the columns for a_1 and a_2 in the tableau above and replace the basic variables x_2 and x_3 in the objective function of the original LP:

$$z = x_1 - 2x_2 + 3x_3$$

$$x_2 = \frac{12}{5} - \frac{14}{5}x_1 - \frac{1}{5}x_4 + \frac{4}{5}x_5$$

$$x_3 = \frac{6}{5} + \frac{13}{5}x_1 + \frac{2}{5}x_4 - \frac{3}{5}x_5$$

$$z = x_1 - 2x_2 + 3x_3 = -\frac{6}{5} + \frac{72}{5}x_1 + \frac{8}{5}x_4 - \frac{17}{5}x_5$$

THE TWO-PHASE SIMPLEX METHOD EXAMPLE

Step 0 Tableau (Phase 2) (after eliminating x_2 and x_3 from the z-row):

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs	Basis
1	$-\frac{72}{5}$	0	0	$-\frac{8}{5}$	$\frac{17}{5}$	0	$-\frac{6}{5}$	z
0	$-\frac{13}{5}$	0	1	$-\frac{2}{5}$	$\frac{3}{5}$	0	$\frac{6}{5}$	x_3
0	$\frac{14}{5}$	1	0	$\frac{1}{5}$	$-\frac{4}{5}$	0	$\frac{12}{5}$	x_2
0	$\frac{4}{5}$	0	0	$\frac{1}{5}$	$\frac{1}{5}$	1	$\frac{27}{5}$	x_6

THE TWO-PHASE SIMPLEX METHOD EXAMPLE

Step 1 Tableau (Phase 2) :

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs	Basis
1	0	$\frac{36}{7}$	0	$-\frac{4}{7}$	$-\frac{4}{7}$	0	$\frac{78}{7}$	z
0	0	$\frac{13}{14}$	1	$-\frac{3}{14}$	$-\frac{1}{7}$	0	$\frac{24}{7}$	x_3
0	1	$\frac{5}{14}$	0	$\frac{1}{14}$	$-\frac{2}{7}$	0	$\frac{6}{7}$	x_1
0	0	$\frac{-2}{7}$	0	$\frac{1}{7}$	$\frac{3}{7}$	1	$\frac{33}{7}$	x_6

THE TWO-PHASE SIMPLEX METHOD EXAMPLE

Step 2 Tableau (Phase 2) :

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs	Basis
1	0	$\frac{14}{3}$	0	$-\frac{1}{3}$	0	$\frac{5}{3}$	19	z
0	0	$\frac{5}{6}$	1	$-\frac{1}{6}$	0	$\frac{1}{3}$	5	x_3
0	1	$\frac{1}{6}$	0	$\frac{1}{6}$	0	$\frac{2}{3}$	4	x_1
0	0	$\frac{-2}{3}$	0	$\frac{1}{3}$	1	$\frac{7}{3}$	11	x_4

THE TWO-PHASE SIMPLEX METHOD EXAMPLE

Step 3 Tableau (Phase 2) : optimal

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs	Basis
1	2	5	0	0	0	3	27	z
0	1	1	1	0	0	1	9	x_3
0	6	1	0	*1*	0	4	24	x_4
0	- 2	- 1	0	0	1	1	3	x_5

The optimal solution : $x_1^* = x_2^* = 0;\ x_3^* = 9;\ z^* \ = 27$

Thank you!