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# SIMPLEX METHOD: INTRODUCTION

LECTURE 6/SURVEY OF OPTIMIZATION

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# AGENDA

- Major Methods of Solving LP
- Characteristics of LPs standard form
- Converting an LP to standard form
- Converting Constraints to Equalities
- Dictionary Format of LP Standard Form
- The Simplex Method
- Heavenly Pouch Inc. Problem : Simplex Method

# MAJOR METHODS OF SOLVING LP

- The Graphical Method ( for 2 variables)
- The Substitution Method
- The Linear Combination Method
- Simplex Method
- Khachiyan's Algorithm (polynomial-time solvability of linear programs)
- The ellipsoid methods (N. Shor's methods)
- Karmarkar's algorithm (an interior-point method)

# CHARACTERISTICS OF LPs STANDARD FORM

- all variables involved are restricted to be non-negative;
- all constraints are equalities, with constant, non-negative right-hand sides;

# CONVERTING an LP TO STANDARD FORM

Converting may require new variables and rearranging constraints:

- an inequality can be multiplied by $-1$ to get non-negative rhs;
- inequalities can be converted to equalities by adding or subtracting non negative slack variables;
- Unrestricted variables can be dealt with by writing the variable as the difference of two new non-negative variables;

# EXAMPLE 1: CONVERTING an LP TO STANDARD FORM

Minimize $80x + 60y$

subject to $x + y \geq 1$

$-0.05x + 0.07y \leq 0$

$x, y \geq 0,$ (non-negativity constraints)

# EXAMPLE 1: CONVERTING an LP TO STANDARD FORM

**STEP 1:** introduce two new variables, $s_1 \geq 0$ and $s_2 \geq 0$ such that:

$s_1$ measures how much over 1 the quantity $x + y$ is;

$s_2$ measures how much under 0 the quantity $-0.05x + 0.07y$ is;

**STEP 2:** update the model accordingly:

$$
\begin{aligned}
\text{Minimize} \quad & 80x + 60y \\
\text{subject to} \quad & x + y - s_1 = 1 \\
& -0.05x + 0.07y + s_2 = 0 \\
& x, y, s_1, s_2 \geq 0
\end{aligned}
$$

# EXAMPLE 1: COMMENTS

- if $(x, y, s_1, s_2)$ is feasible for the updated model, then $(x, y)$ is feasible for the original model;
- if $(x, y)$ is feasible for the original, then $(x, y, (x + y) - 1, 0 - (-0.05x + 0.07y))$ is feasible for the updated problem;
- since the objective only involves x and y, the two problems have the same solution;

# CONVERTING CONSTRAINTS TO EQUALITIES

➢ If a constraint is of "≥" type , we introduce a nonnegative *excess variable* $e_j$ and subtract it from the left-hand side of the constraint to obtain an equality:

Ex. $x_1 + 2x_2 + 3x_3 \geq 4$ ⟶ $x_1 + 2x_2 + 3x_3 - e_j = 4,$

where $e_j = -4 + x_1 + 2x_2 + 3x_3 \geq 0.$

# CONVERTING CONSTRAINTS TO EQUALITIES

- If a constraint is of "≤" type , we add a nonnegative *slack variable* $s_i$ in the left-hand side of the constraint to turn it into an equality:

Ex. $x_1 + 2x_2 + 3x_3 \leq 4 \longrightarrow x_1 + 2x_2 + 3x_3 + s_i = 4,$

where $s_i = 4 - x_1 - 2x_2 - 3x_3 \geq 0.$

# EXAMPLE 2: CONVERTING an LP TO STANDARD FORM

Original Form of the LP:

$$\text{maximize} \quad 3x_1 - 5x_2 + 7x_3$$

$$\begin{aligned}\text{subject to} \quad & 2x_1 + 4x_2 - x_3 \geq -3 \\ & 4x_1 - 2x_2 + 8x_3 \leq 7 \\ & 9x_1 + x_2 + 3x_3 = 11 \\ & x_1, x_2, x_3 \geq 0\end{aligned}$$

Standard Form of the LP:

$$\text{maximize} \quad 3x_1 - 5x_2 + 7x_3$$

$$\begin{aligned}\text{subject to} \quad & 2x_1 + 4x_2 - x_3 - e_1 = -3 \\ & 4x_1 - 2x_2 + 8x_3 + s_2 = 7 \\ & 9x_1 + x_2 + 3x_3 = 11 \\ & x_1, x_2, x_3, e_1, s_2 \geq 0\end{aligned}$$

# EXAMPLE 3: DICTIONARY FORMAT OF STANDARD FORM LP

Original Form of the LP:

maximize $5x_1 + 5x_2 + 3x_3$

subject to $x_1 + 3x_2 + x_3 \leq 3$

$-x_1 + 3x_3 \leq 2$

$2x_1 - x_2 + 2x_3 \leq 4$

$2x_1 + 3x_2 - x_3 \leq 2$

$x_1, x_2, x_3 \geq 0$

Dictionary Format of the LP:

$$z = 5x_1 + 5x_2 + 3x_3$$

$$\begin{aligned} s_1 &= 3 - x_1 - 3x_2 - x_3 \\ s_2 &= 2 + x_1 - 3x_3 \\ s_3 &= 4 - 2x_1 + x_2 - 2x_3 \\ s_4 &= 2 - 2x_1 - 3x_2 + x_3 \end{aligned}$$

$s_1, s_2, s_3, s_4$ are slack variables.

# EXAMPLE 3: DICTIONARY FORMAT OF STANDARD FORM LP

We do not explicitly state this, but we assume that all the variables are nonnegative in the dictionary format.

Rename the slack variables:

$x_4 = s_1$ , $x_5 = s_2$ , $x_6 = s_3$ , $x_7 = s_4$

Basic variables: $x_4$ ,$x_5$ ,$x_6$ ,$x_7$
(Basis B = { 4,5,6,7 })

Non-basic variables: $x_1$ ,$x_2$ ,$x_3$
(N = {1,2,3})

Dictionary Format of the LP:

$$z = 5x_1 + 5x_2 + 3x_3$$

$$\begin{aligned}
x_4 &= 3 - x_1 - 3x_2 - x_3 \\
x_5 &= 2 + x_1 - 3x_3 \\
x_6 &= 4 - 2x_1 + x_2 - 2x_3 \\
x_7 &= 2 - 2x_1 - 3x_2 + x_3
\end{aligned}$$

# THE SIMPLEX METHOD

**Step 0:** Initialization: Basic Feasible Solution

If all variables have nonnegative values in a basic solution, then the solution is called *a basic feasible solution (bfs)* and the corresponding *dictionary* is called *feasible*.

$$z = 5x_1 + 5x_2 + 3x_3$$

$$x_4 = 3 - x_1 - 3x_2 - x_3$$
$$x_5 = 2 + x_1 - 3x_3$$
$$x_6 = 4 - 2x_1 + x_2 - 2x_3$$
$$x_7 = 2 - 2x_1 - 3x_2 + x_3$$

- Basic variables: $x_4, x_5, x_6, x_7$ (Basis B = { 4,5,6,7 })
- Non-basic variables: $x_1, x_2, x_3$ (N = {1,2,3})
- The initial bfs: $x_1 = x_2 = x_3 = 0$

  $x_4 = 3, x_5 = 2, x_6 = 4, x_7 = 2;\ z = 0$

# THE SIMPLEX METHOD

- ❑ Iterative Improvement: Pivot Variable (Column)

In order to increase the value of the objective function z in the feasible dictionary , increase the value of one of the non-basic variables with a maximum positive coefficient in the objective. Then, take a variable with the largest coefficient in zero-row, which is called the pivot variable and the corresponding column in the table is called the pivot column.

# THE SIMPLEX METHOD

- Determining the Pivot Row: Ratio Test

Increase the value of $x_1$ (it has the largest positive coefficient in the objective of 5) while the remaining nonbasic variables remain equal to 0.

$$x_4 = 3 - x_1 \geq 0$$

$$x_5 = 2 + x_1 \geq 0$$

$$x_6 = 4 - 2x_1 \geq 0$$

$$x_7 = 2 - 2x_1 \geq 0$$

In order to satisfy the inequalities $x_1 \leq 1$ ➡

➡ the largest for $x_1$ is 1 ➡

➡ the largest feasible increase for $x_1$ is 1

# THE SIMPLEX METHOD

- Determining the Pivot Row: Ratio Test

The largest possible increase corresponds to the smallest ratio of the free coefficient to the absolute value of the coefficient for $x_1$ in the same row, assuming that the coefficient for $x_1$ is negative. We say that the row in which the *smallest ratio* is achieved *wins the ratio test*. This row is called *the pivot row*.

Note that the minimum ratio gives the value of $x_1$ as it enters the basis.

# THE SIMPLEX METHOD Performing the Pivot:

Feasible Dictionary obtained at Step 0:

pivot column

$$z = 5x_1 + 5x_2 + 3x_3$$

$$x_4 = 3 - x_1 - 3x_2 - x_3$$
$$x_5 = 2 + x_1 - 3x_3$$
$$x_6 = 4 - 2x_1 + x_2 - 2x_3$$
$$x_7 = 2 - 2x_1 - 3x_2 + x_3$$

pivot row

$x_1$ wins the ratio test :

$$x_1 = 1 - \frac{3}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_7$$

# THE SIMPLEX METHOD Performing the Pivot:

Step 1: substitute this expression for $x_1$ in the remaining rows of the dictionary

$$z = 5x_1 - \frac{5}{2}x_2 + \frac{11}{2}x_3 - \frac{5}{2}x_7$$

$$x_1 = 1 - \frac{3}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_7$$

$$x_4 = 2 - \frac{3}{2}x_2 - \frac{3}{2}x_3 + \frac{1}{2}x_7$$

$$x_5 = 3 - \frac{3}{2}x_2 - \frac{5}{2}x_3 - \frac{1}{2}x_7$$

$$x_6 = 2 + 4x_2 - 3x_3 + x_7$$

there is still potential for improvement by increasing the value of $x_3$

- Basic variables: $x_1, x_4, x_5, x_6$ ($B = \{1,4,5,6\}$)
- Non-basic variables: $x_2, x_3, x_7$ ($N = \{2,3,7\}$).
- Step 1 bfs: $x_2 = x_3 = x_7 = 0$;
  $x_1 = 1, x_4 = 2, x_5 = 3, x_6 = 2$; $z = 5$.

# THE SIMPLEX METHOD    Performing the Pivot:

**Step 2:** $x_3$ – the pivot variable (entering variable); the last row wins the test with the ratio of 2/3 (min (4/3, 6/5, 2/3))

$$z = 5x_1 - \frac{5}{2}x_2 + \frac{11}{2}x_3 - \frac{5}{2}x_7$$

$$x_1 = 1 - \frac{3}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_7$$

$$x_4 = 2 - \frac{3}{2}x_2 - \frac{3}{2}x_3 + \frac{1}{2}x_7$$

$$x_5 = 3 - \frac{3}{2}x_2 - \frac{5}{2}x_3 - \frac{1}{2}x_7$$

$$x_6 = 2 + 4x_2 - 3x_3 + x_7$$

there is still potential for improvement by increasing the value of $x_3$

$$x_3 = \frac{2}{3} + \frac{4}{3}x_2 + \frac{1}{3}x_7 - \frac{1}{3}x_6$$

Substitute: *(next slide)*

# THE SIMPLEX METHOD

## Performing the Pivot:

$$z = \frac{26}{3}x_1 + \frac{29}{6}x_2 - \frac{2}{3}x_7 - \frac{11}{6}x_6$$

$$x_3 = \frac{2}{3} + \frac{4}{3}x_2 + \frac{1}{3}x_7 - \frac{1}{3}x_6$$

$$x_1 = \frac{4}{3} - \frac{5}{6}x_2 - \frac{1}{2}x_7 - \frac{1}{6}x_6$$

$$x_4 = 1 - \frac{7}{2}x_2 + \frac{1}{2}x_6$$

$$x_5 = \frac{4}{3} - \frac{29}{6}x_2 - \frac{4}{3}x_7 + \frac{5}{6}x_6$$

there is still potential for improvement by increasing the value of $x_2$

- Basic variables: $x_1, x_4, x_5, x_3$ (B = { 1,3,4,5 } )
- Non-basic variables: $x_2, x_6, x_7$ (N = { 2,6,7 } ).
- Step 2 bfs: $x_2 = x_6 = x_7 = 0$;

  $x_1 = \frac{4}{3}, x_3 = \frac{2}{3}, x_4 = 1, x_5 = \frac{4}{3}; z = \frac{26}{3}$

# THE SIMPLEX METHOD — Performing the Pivot:

**Step 3:** $x_2$ – the pivot variable (entering variable); row 4 wins the test with the ratio of 8/29 (min (8/5, 2/7, 8/29))

$$z = \frac{26}{3}x_1 + \frac{29}{6}x_2 - \frac{2}{3}x_7 - \frac{11}{6}x_6$$

$$x_3 = \frac{2}{3} + \frac{4}{3}x_2 + \frac{1}{3}x_7 - \frac{1}{3}x_6$$

$$x_1 = \frac{4}{3} - \frac{5}{6}x_2 - \frac{1}{2}x_7 - \frac{1}{6}x_6$$

$$x_4 = 1 - \frac{7}{2}x_2 + \frac{1}{2}x_6$$

$$x_5 = \frac{4}{3} - \frac{29}{6}x_2 - \frac{4}{3}x_7 + \frac{5}{6}x_6$$

there is still potential for improvement by increasing the value of $x_2$

$$x_2 = \frac{8}{29} - \frac{8}{29}x_7 + \frac{5}{29}x_6 - \frac{6}{29}x_5$$

Substitute: *(next slide)*

# THE SIMPLEX METHOD

## Performing the Pivot:

$$z = 10 - 2x_2 - x_6 - x_5$$

$$x_2 = \frac{8}{29} - \frac{8}{29}x_7 + \frac{5}{29}x_6 - \frac{6}{29}x_5$$

$$x_3 = \frac{30}{29} - \frac{1}{29}x_7 - \frac{3}{29}x_6 - \frac{8}{29}x_5$$

$$x_1 = \frac{32}{29} - \frac{3}{29}x_7 - \frac{9}{29}x_6 + \frac{5}{29}x_5$$

$$x_4 = \frac{1}{29} + \frac{28}{29}x_7 - \frac{3}{29}x_6 + \frac{21}{29}x_5$$

- Basic variables: $x_1, x_2, x_3, x_4$ (B = { 1,2,3,4 } )
- Non-basic variables: $x_5, x_6, x_7$ (N = { 5,6,7 } ).
- Step 3 bfs: $x_5 = x_6 = x_7 = 0$;

$$x_1 = \frac{32}{29}, x_2 = \frac{8}{29}, x_3 = \frac{30}{29}, x_4 = \frac{1}{29}; z = \mathbf{10}$$

optimal

# HEAVENLY POUCH INC. PROBLEM

| | | |
|---|---|---|
| Maximize | $15x_1 + 25x_2$ | (profit) |
| Subject to (s.t.) | $x_1 + x_2 \leq 450$ | (solid color fabric constraint) |
| | $x_2 \leq 300$ | (printed fabric constraint) |
| | $4x_1 + 5x_2 \leq 2000$ | (budget constraint) |
| | $x_1 \leq 350$ | (demand constraint) |
| | $x_1, x_2 \geq 0$ | (nonnegativity constraints) |

# HEAVENLY POUCH INC. PROBLEM: SIMPLEX METHOD

Convert the LP to the standard form by introducing a slack variable $s_i$ for each constraint i, i = 1,...,4:

$$
\begin{aligned}
\text{Maximize} \quad & 15x_1 + 25x_2 \\
\text{Subject to (s.t.)} \quad & x_1 + x_2 + s_1 = 450 \\
& x_2 + s_2 = 300 \\
& 4x_1 + 5x_2 + s_3 = 2000 \\
& x_1 + s_4 = 350 \\
& x_1, x_2, s_1, s_2, s_3, s_4 \geq 0
\end{aligned}
$$

# HEAVENLY POUCH INC. PROBLEM: SIMPLEX METHOD

## Dictionary format

$$z = \qquad 15x_1 + 25x_2$$

$$s_1 = 450 \quad - x_1 \quad - x_2$$
$$s_2 = 300 \qquad\qquad - x_2$$
$$s_3 = 2{,}000 - 4x_1 - 5x_2$$
$$s_4 = 350 \quad - x_1$$

Set of basic variables: $BV_0 = \{ s_1, s_2, s_3, s_4 \}$

Set of nonbasic variables: $NV_0 = \{ x_1, x_2 \}$

# HEAVENLY POUCH INC. PROBLEM: SIMPLEX METHOD

Set all the nonbasic variables to 0:

$$x_1 = x_2 = 0 \Rightarrow s_1 = 450, s_2 = 300, s_3 = 2{,}000, s_4 = 350; \ z = 0$$

LP in the *tableau format*:
(z is modified by moving all the variables to the left:
$z - 15x_1 - 25x_2 = 0$)

| z | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | rhs | Basis |
|---|---|---|---|---|---|---|---|---|
| 1 | -15 | -25 | 0 | 0 | 0 | 0 | 0 | z |
| 0 | 1 | 1 | 1 | 0 | 0 | 0 | 450 | $s_1$ |
| 0 | 0 | 1 | 0 | 1 | 0 | 0 | 300 | $s_2$ |
| 0 | 4 | 5 | 0 | 0 | 1 | 0 | 2000 | $s_3$ |
| 0 | 1 | 0 | 0 | 0 | 0 | 1 | 350 | $s_4$ |

# HEAVENLY POUCH INC. PROBLEM: SIMPLEX METHOD

| **Step 0 basic feasible solution** | |
|---|---|
| $BV_0$ : | $s_1, s_2, s_3, s_4$ |
| $NV_0$ : | $x_1, x_2$ |
| b f s : | $x_1 = x_2 = 0$<br>$s_1 = 450, s_2 = 300, s_3 = 2{,}000, s_4 = 350$<br>$z = 0$ |

# HEAVENLY POUCH INC. PROBLEM: SIMPLEX METHOD

Perform the first iteration of the simplex method

Step 1

$$z = 15x_1 + 25x_2$$

Pivot variable

$$s_1 = 450 - x_1 - x_2$$
$$s_2 = 300 - x_2$$
$$s_3 = 2000 - 4x_1 - 5x_2$$
$$s_4 = 300 - x_1$$

Pivot row

(Min (450/1; 300/1; 2000/5))

Substitute $x_2 = 300 - s_2$ *(next slide)*

# HEAVENLY POUCH INC. PROBLEM: SIMPLEX METHOD

$$z = 15x_1 + 25x_2 = 15x_1 + 25(300 - s_2) = 7{,}500 + 15x_1 - 25s_2$$

---

$$s_1 = 450 - x_1 - x_2 = 450 - x_1 - (300 - s_2) = 150 - x_1 + s_2$$

$$s_3 = 2{,}000 - 4x_1 - 5x_2 = 2{,}000 - 4x_1 - 5(300 - s_2) = 500 - 4x_1 + 5s_2$$

$$s_4 = 350 - x_1 - 0x_2 = 350 - x_1$$

# HEAVENLY POUCH INC. PROBLEM: SIMPLEX METHOD

Tableau :

| z | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | rhs | Basis | Ratio |
|---|---|---|---|---|---|---|---|---|---|
| 1 | -15 | -25 | 0 | 0 | 0 | 0 | 0 | z | |
| 0 | 1 | 1 | 1 | 0 | 0 | 0 | 450 | $s_1$ | 450 |
| 0 | 0 | 1 | 0 | 1 | 0 | 0 | 300 | $s_2$ | 300 |
| 0 | 4 | 5 | 0 | 0 | 1 | 0 | 2000 | $s_3$ | 400 |
| 0 | 1 | 0 | 0 | 0 | 0 | 1 | 350 | $s_3$ | --- |

# HEAVENLY POUCH INC. PROBLEM: SIMPLEX METHOD

## Step 1 Dictionary:

Pivot variable

$$z = 7{,}500 + 15x_1 - 25s_2$$

$$s_1 = 150 - x_1 + s_2$$

$$x_2 = 300 - s_2$$

$$s_3 = 500 - 4x_1 + 5s_2$$

$$s_4 = 350 - x_1$$

## Step 1 Tableau:

| z | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | rhs | Basis | Ratio |
|---|---|---|---|---|---|---|---|---|---|
| 1 | -15 | 0 | 0 | -25 | 0 | 0 | 7,500 | z | |
| 0 | 1 | 0 | 1 | -1 | 0 | 0 | 150 | $s_1$ | 150 |
| 0 | 0 | 1 | 0 | 1 | 0 | 0 | 300 | $x_2$ | ------ |
| 0 | 4 | 0 | 0 | -5 | 1 | 0 | 500 | $s_3$ | 125 |
| 0 | 1 | 0 | 0 | 0 | 0 | 1 | 350 | $s_4$ | 350 |

Row 3 wins

# HEAVENLY POUCH INC. PROBLEM: SIMPLEX METHOD

| Step 1 basic feasible solution | |
|---|---|
| $BV_1$ : | $s_1, x_2, s_3, s_4$ |
| $NV_1$ : | $x_1, s_2$ |
| b f s : | $x_1 = 0, x_2 = 300$<br>$s_1 = 150, s_2 = 0, s_3 = 500, s_4 = 350$<br>$z = 7{,}500$ |

# HEAVENLY POUCH INC. PROBLEM: SIMPLEX METHOD

Step 2: Row 3 wins the ratio test, so $s_3$ is the *leaving variable* :

$$x_1 = 125 + \frac{5}{4}s_2 - \frac{1}{4}s_3$$

*Substitute* →

$$z = 7{,}500 + 15x_1 - 25s_2 = 9{,}375 - \frac{25}{4}s_2 - \frac{15}{4}s_3$$

$$s_1 = 150 - x_1 + s_2 = 25 - \frac{1}{4}s_2 + \frac{1}{4}s_3$$

$$s_4 = 350 - x_1 = 225 - \frac{1}{4}s_2 + \frac{1}{4}s_3$$

# HEAVENLY POUCH INC. PROBLEM: SIMPLEX METHOD

## Step 2 Dictionary:

$$z = 9{,}375 - \frac{25}{4}s_2 - \frac{15}{4}s_3$$

$$s_1 = 25 - \frac{1}{4}s_2 + \frac{1}{4}s_3$$

$$x_2 = 300 - s_2$$

$$x_1 = 125 + \frac{5}{4}s_2 - \frac{1}{4}s_3$$

$$s_4 = 225 - \frac{5}{4}s_2 + \frac{1}{4}s_3$$

## Step 2 Tableau:

| z | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | rhs | Basis | Ratio |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 25/4 | 15/4 | 0 | 9,375 | z | |
| 0 | 0 | 0 | 1 | 1/4 | -1/4 | 0 | 25 | $s_1$ | |
| 0 | 0 | 1 | 0 | 1 | 0 | 0 | 300 | $x_2$ | |
| 0 | 1 | 0 | 0 | -5/4 | 1/4 | 0 | 125 | $x_1$ | |
| 0 | 0 | 0 | 0 | 5/4 | -1/4 | 1 | 225 | $s_4$ | |

# HEAVENLY POUCH INC. PROBLEM: SIMPLEX METHOD

| **Step 2 basic feasible solution** | |
|---|---|
| $BV_2$ : | $x_1, x_2, s_1, s_4$ |
| $NV_2$ : | $s_3, s_2$ |
| b f s : | $x_1 = 125; x_2 = 300$<br>$s_1 = 25, s_2 = 0, s_3 = 0, s_4 = 225$<br>$z = 9{,}375$ |

# Thank you!