

V.M. GLUSHKOV INSTITUTE
OF CYBERNETICS OF THE NAS OF UKRAINE

LP ASSUMPTIONS AND THE GRAPHICAL METHOD

LECTURE 5/SURVEY OF OPTIMIZATION

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AGENDA

- Major Methods of Solving LP
- A Production Planning Problem
- Practical Implications of Using LP Models
- Heavenly Pouch Inc. Problem : Graphical Solution
- Advertising campaign planning: Graphical Solution
- Classification of LPs

MAJOR METHODS OF SOLVING LP

- The Graphical Method (for 2 variables)
- The Substitution Method
- The Linear Combination Method
- Simplex Method
- Khachiyan's Algorithm (polynomial-time solvability of linear programs)
- The ellipsoid methods (N. Shor's methods)
- Karmarkar's algorithm (an interior-point method)

A PRODUCTION PLANNING PROBLEM

PRODUCTION PLANNING



MIA Corporation manufactures n products P_1, \dots, P_n using m different types of resources (such as raw material, labor, etc.), R_1, \dots, R_m . There are b_i units of resource i available per week. Manufacturing one unit of product P_j requires a_{ij} units of resource R_i . It is known that each unit of product P_j will sell for c_j dollars. MIA Corporation needs to decide how many units of each product to manufacture in order to maximize its weekly profit.



A PRODUCTION PLANNING PROBLEM (cont.)

Step 1: Define the decision variables:

x_j = the number of units of product j to manufacture, $j = 1, \dots, n$.

Step 2: State the objective function:

maximize $z: c_1x_1 + \dots + c_nx_n$ (maximize weekly profit)



A PRODUCTION PLANNING PROBLEM (cont.)

Step 3: Specify the *constraints*

the corporation does not exceed the availability of each resource

$$a_{i1}x_1 + \dots + a_{in}x_n \leq b_i, i = 1, 2, \dots, m \quad (\text{the resource constraints})$$

$$x_1, \dots, x_n \geq 0, \quad (\text{non-negativity constraints})$$



A PRODUCTION PLANNING PROBLEM (cont.)

Final Model Formulation

maximize $c_1x_1 + \dots + c_nx_n$ (maximize weekly profit)

subject to $a_{11}x_1 + \dots + a_{1n}x_n \leq b_1$
 \vdots
 \vdots
 \vdots (the resources constraints)

$$a_{m1}x_1 + \dots + a_{mn}x_n \leq b_m$$

$x_1, \dots, x_n \geq 0$ (non-negativity constraints)

VECTORS AND MATRICES

- A **real n-dimensional vector** is an ordered set of n real numbers $\{x_1, x_2, \dots, x_n\}$ and is usually written in the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix} \in \mathbb{R}^n \quad \text{or } \mathbf{x} = [x_1, x_2, \dots, x_n].$$

(column vector)

(row vector)

- Another form: $\mathbf{x} = [x_j]_{j=1}^n$. The numbers x_1, x_2, \dots, x_n are called the **components** of \mathbf{x} .
- For clarity, unless otherwise specified, by a vector we will mean the column vector.

VECTORS AND MATRICES

- A **real matrix** is a rectangular array of real numbers composed of rows and columns.

$$A = [a_{ij}]_{m \times n} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \dots & \cdot & \dots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m \times n}$$

for a matrix of m rows and n columns, and we say that the matrix A is of order $m \times n$.

- An $m \times n$ matrix has m rows and n columns.
- Given a matrix $A \in \mathbb{R}^{m \times n}$, the transpose of A is $n \times m$ matrix whose rows are the columns of A , denoted by A^T .



A PRODUCTION PLANNING PROBLEM (cont.)

Matrix Form of the LP Model

Denote:

$$A = \begin{bmatrix} a_{11} & \cdot & a_{1n} \\ \cdot & \cdot & \cdot \\ a_{m1} & \cdot & a_{mn} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ \cdot \\ b_m \end{bmatrix}, \quad c = \begin{bmatrix} c_1 \\ \cdot \\ c_n \end{bmatrix}, \quad x = [x_1, x_2, \dots, x_n]^T$$

$$\text{Maximize } c^T x$$

$$\text{Subject to } Ax \leq b$$

$$x \geq 0$$

$$\text{where } c = [c_j]_{j=1}^n, \quad b = [b_i]_{i=1}^m$$

$$A = [a_{ij}]_{m \times n}, \quad x = [x_j]_{j=1}^n$$



A PRODUCTION PLANNING PROBLEM (cont.)

Equivalent Form of the Model Formulation

maximize
$$\sum_{j=1}^n C_j x_j$$

subject to
$$\sum_{j=1}^n a_{ij} x_j \leq b_i, i=1, \dots, m$$

$$x_j \geq 0, j = 1, \dots, n$$

PRACTICAL IMPLICATIONS OF USING LP MODELS

ADVANTAGES	DISADVANTAGES
- A relative simplicity	- Restricting assumptions
- Scalability of algorithms	linearity of dependencies
- Implementability in software packages	certainty of dependencies
	- Most real-life processes are non-linear in nature

LINEAR FUNCTIONS PROPERTIES

A linear function $f(x)$ in \mathbb{R}^n is given by $f(x) = c_1x_1 + \dots + c_nx_n$,

where c_1, \dots, c_n are constant real coefficients

- The property of **additivity**: $f(x+y) = f(x) + f(y)$ for any $x, y \in \mathbb{R}^n$,
- The property of **proportionality**: $f(\alpha x) = \alpha f(x)$ for any $x \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$

ASSUMPTIONS OF LINEAR PROGRAMMING

- **Proportionality assumption:** Contribution of a variable is proportional to its value.
- **Additivity assumption:** Contributions of variables are independent.
- **Divisibility assumption:** Decision variables can take fractional values.
- **Certainty assumption:** Each parameter is known with certainty.

Example: $7x_1 + 3x_2$

HEAVENLY POUCH Inc. PROBLEM

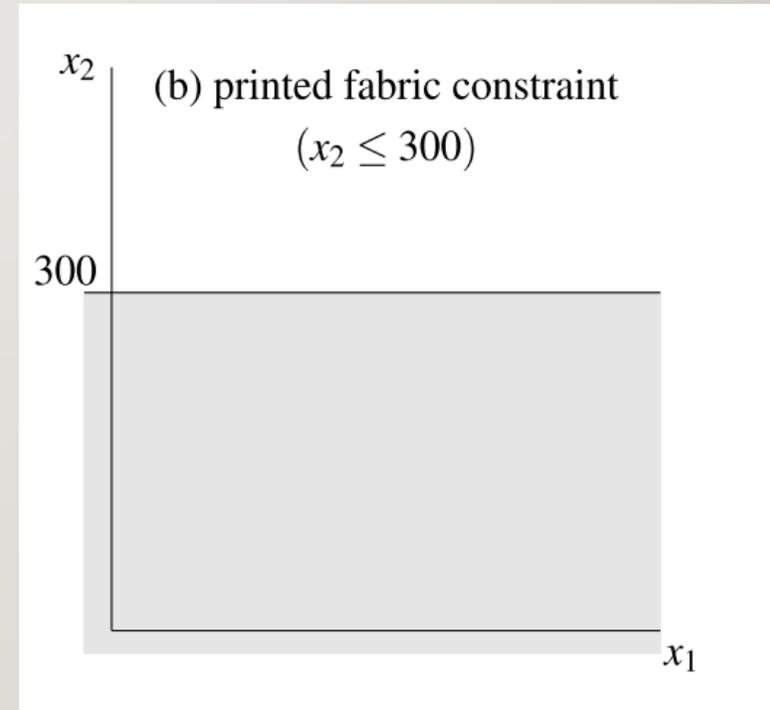
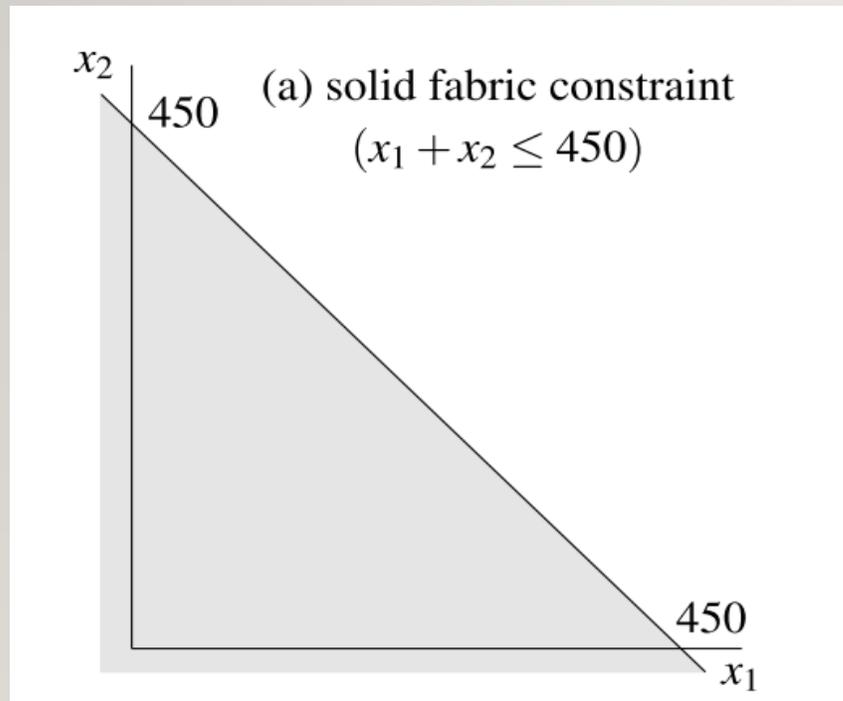
Heavenly Pouch, Inc. produces two types of baby carriers, non-reversible and reversible. Each non-reversible carrier sells for \$23, requires 2 meters of a solid color fabric, and costs \$8 to manufacture. Each reversible carrier sells for \$35, requires 2 meters of a printed fabric as well as 2 meters of a solid color fabric, and costs \$10 to manufacture. The company has 900 meters of solid color fabrics and 600 meters of printed fabrics available for its new carrier collection. It can spend up to \$4,000 on manufacturing the carriers. The demand is such that all reversible carriers made are projected to sell, whereas at most 350 non-reversible carriers can be sold. Heavenly Pouch is interested in formulating a mathematical model that could be used to maximize its profit (e.g., the difference of revenues and expenses) resulting from manufacturing and selling the new carrier collection.

HEAVENLY POUCH INC. PROBLEM

Maximize	$15x_1 + 25x_2$	(profit)
Subject to (s.t.)	$x_1 + x_2 \leq 450$	(solid color fabric constraint)
	$x_2 \leq 300$	(printed fabric constraint)
	$4x_1 + 5x_2 \leq 2000$	(budget constraint)
	$x_1 \leq 350$	(demand constraint)
	$x_1, x_2 \geq 0$	(nonnegativity constraints)

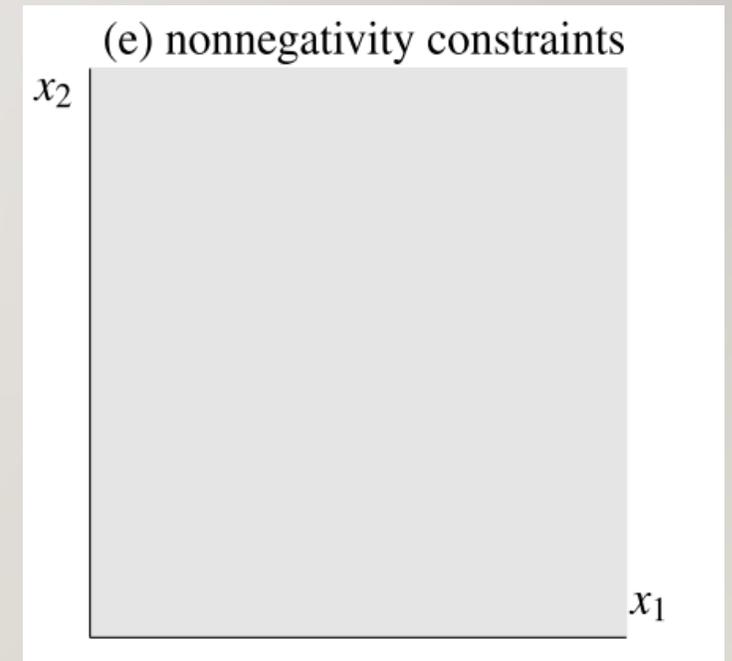
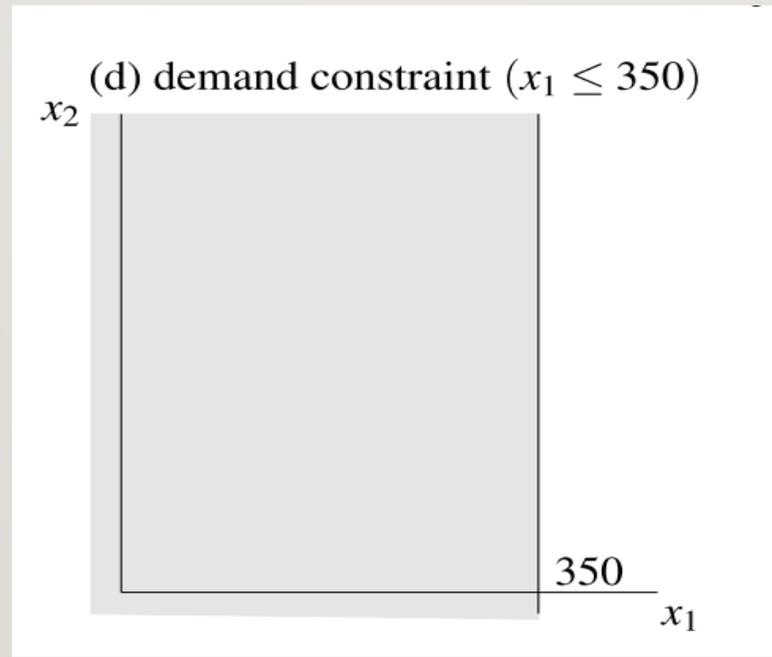
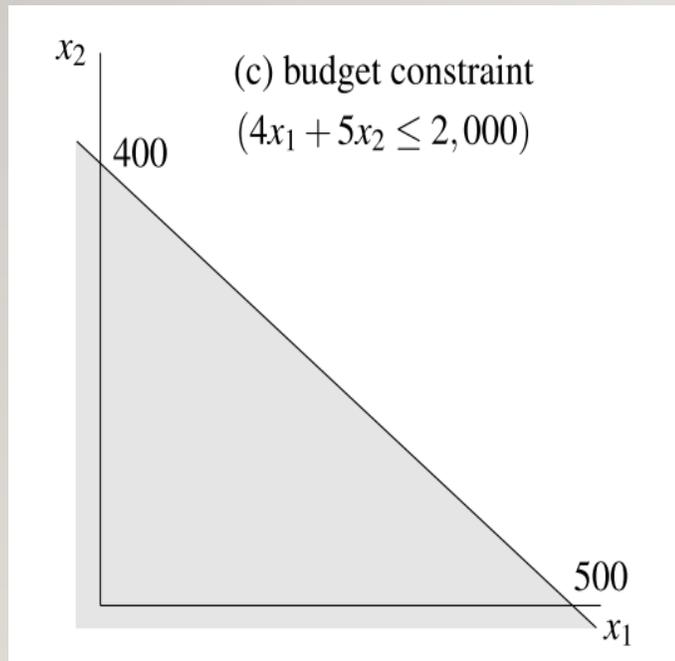
HEAVENLY POUCH INC. PROBLEM SOLUTION

Graph the constraints:



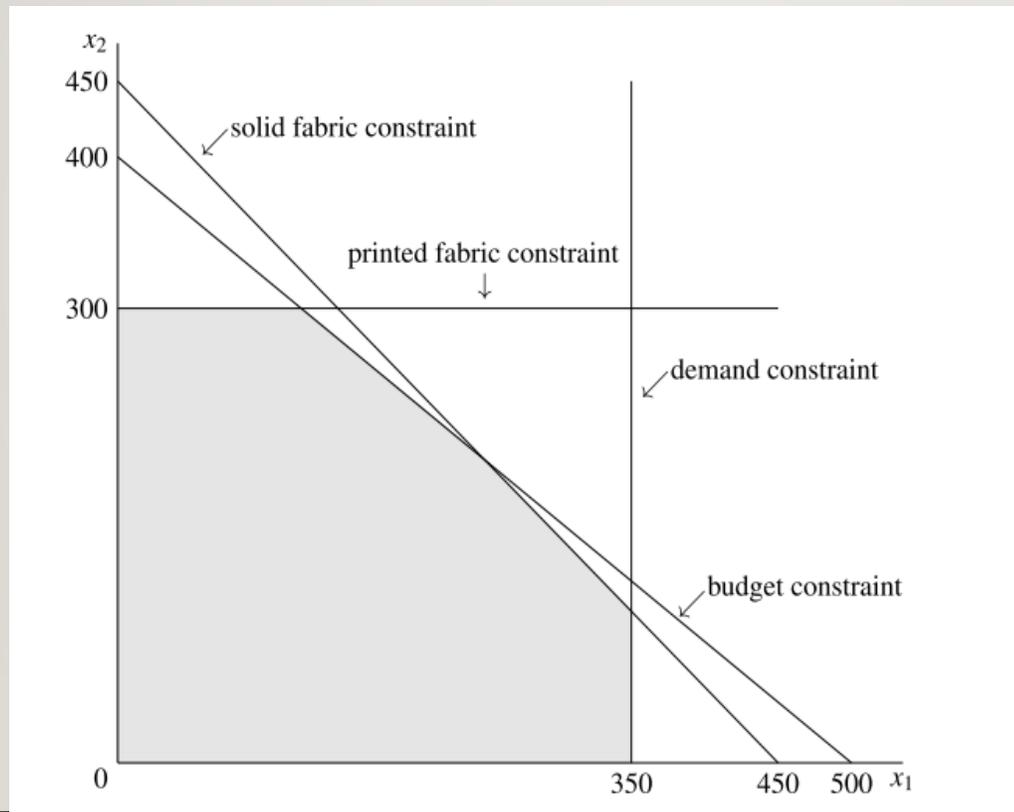
HEAVENLY POUCH INC. PROBLEM

Graph the constraints:



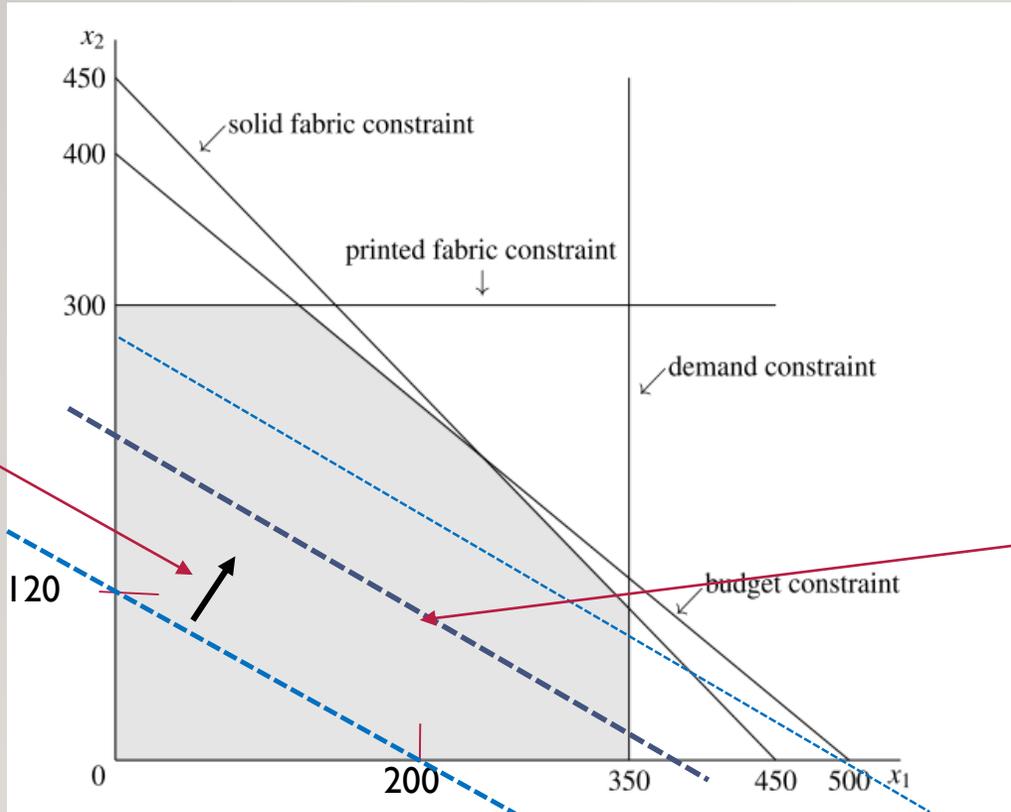
HEAVENLY POUCH INC. PROBLEM

Determine the feasibility region:



HEAVENLY POUCH INC. PROBLEM

g
r
a
d
i
e
n
t



$$\text{Let } z = 15x_1 + 25x_2 = 3,000$$

$$\text{Solution: } x_1 = 0 \quad \longrightarrow \quad 25x_2 = 3,000$$

$$x_2 = 120$$

$$x_2 = 0 \quad \longrightarrow \quad 15x_1 = 3,000$$

$$x_1 = 200$$

“iso”-profit line

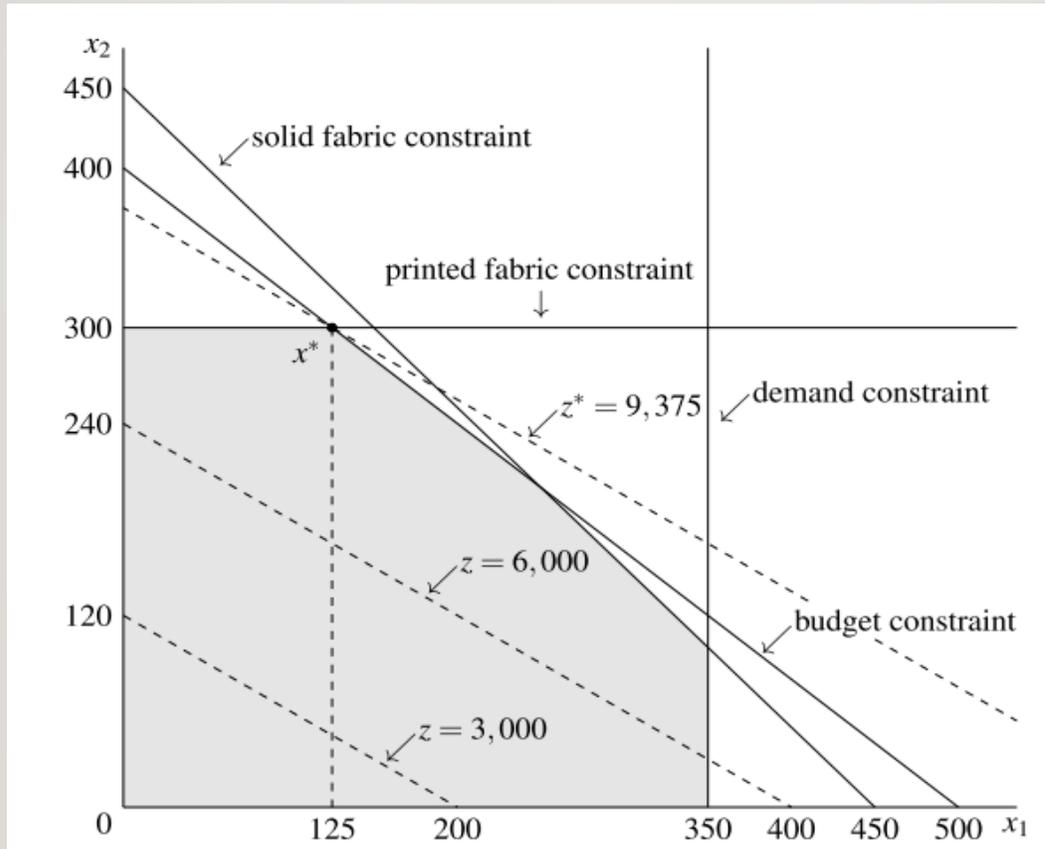
HEAVENLY POUCH INC. PROBLEM

Graphical solution:

X^* lies in intersection of two constraints:

$$x_2 \leq 300 \text{ and} \\ 4x_1 + 5x_2 \leq 2000$$

The intersection point is **(125,300)**



Solution:

$$x_1^* = 125, x_2^* = 300$$

$$z^* = \$15(125) + \\ \$25(300) = \$9,375$$

HEAVENLY POUCH INC. PROBLEM MODIFIED

Suppose that the price of a non-reversible carrier is raised by \$5.

Modified LP model: Maximize $20x_1 + 25x_2$ (profit)

Subject to (s.t.) $x_1 + x_2 \leq 450$ (solid color fabric constraint)

$x_2 \leq 300$ (printed fabric constraint)

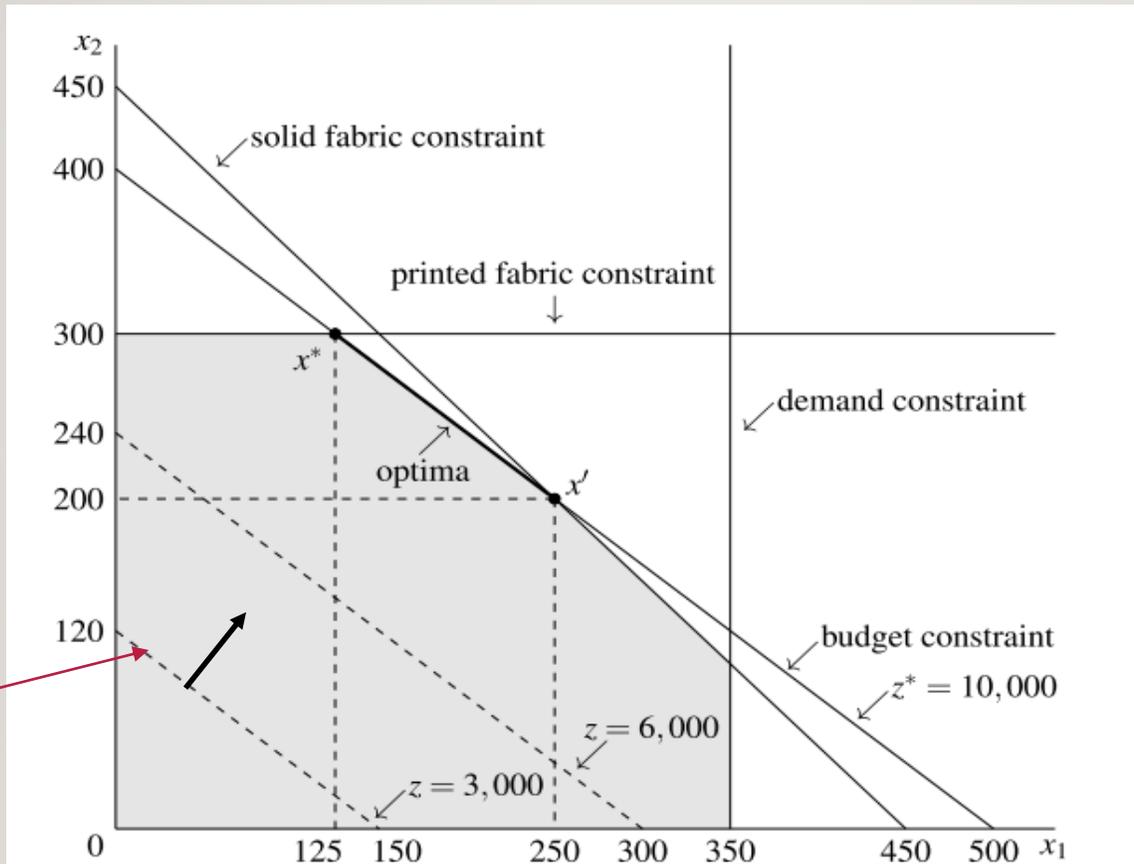
$4x_1 + 5x_2 \leq 2000$ (budget constraint)

$x_1 \leq 350$ (demand constraint)

$x_1, x_2 \geq 0$ (nonnegativity constraints)

HEAVENLY POUCH INC. PROBLEM MODIFIED

Graphical solution:



The slope of the “iso”-profit line is different

Optimal solutions:

All points on the thick line segment between points $x^* = (125, 300)$ and $x' = (250, 200)$

The optimal **objective function**

$$z^* = 10,000$$

ADVERTISING CAMPAIGN PLANNING



A retail store is planning an advertising campaign aiming to increase the number of customers visiting its physical location, as well as its online store. The store manager would like to advertise through a local magazine and through an online social network. She estimates that each 1,000 dollars invested in magazine ads will attract 100 new customers to the store, as well as 500 new website visitors. In addition, each 1,000 dollars invested in online advertising will attract 50 new local store customers, as well as 1,000 new website visitors. Her target for this campaign is to bring at least 500 new guests to the physical store and at least 5,000 new visitors to the online store.

Formulate an LP to help the store minimize the cost of its advertising campaign and solve it graphically.

ADVERTISING CAMPAIGN PLANNING



The decision variables:

x_1 = budget for magazine advertising (in thousands of dollars),

x_2 = budget for online advertising (in thousands of dollars),



ADVERTISING CAMPAIGN PLANNING

Final LP model:

minimize $x_1 + x_2$

subject to $100x_1 + 50x_2 \geq 500$ (store visitors)

$500x_1 + 1,000x_2 \geq 5,000$ (website visitors)

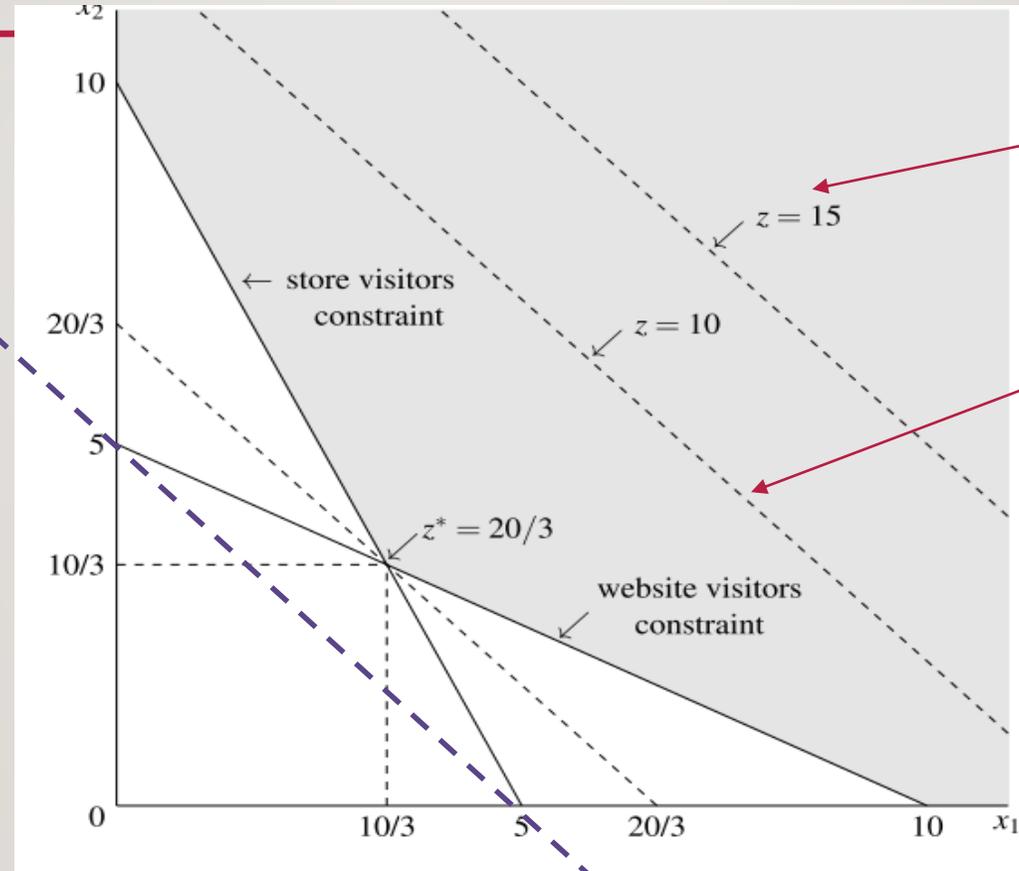
$x_1, x_2 \geq 0$ (nonnegativity)



ADVERTISING CAMPAIGN PLANNING

Graphical solution:

Solve the system:



Feasibility region

ISO-cost line (minimization)



ADVERTISING CAMPAIGN PLANNING

Solve the system:
$$\begin{cases} 100x_1 + 50x_2 = 500 \\ 500x_1 + 1,000x_2 = 5,000 \end{cases}$$

$$x_1^* = x_2^* = 10/3 \approx 3.3333 \qquad z^* = 20/3 \approx 6.66666$$

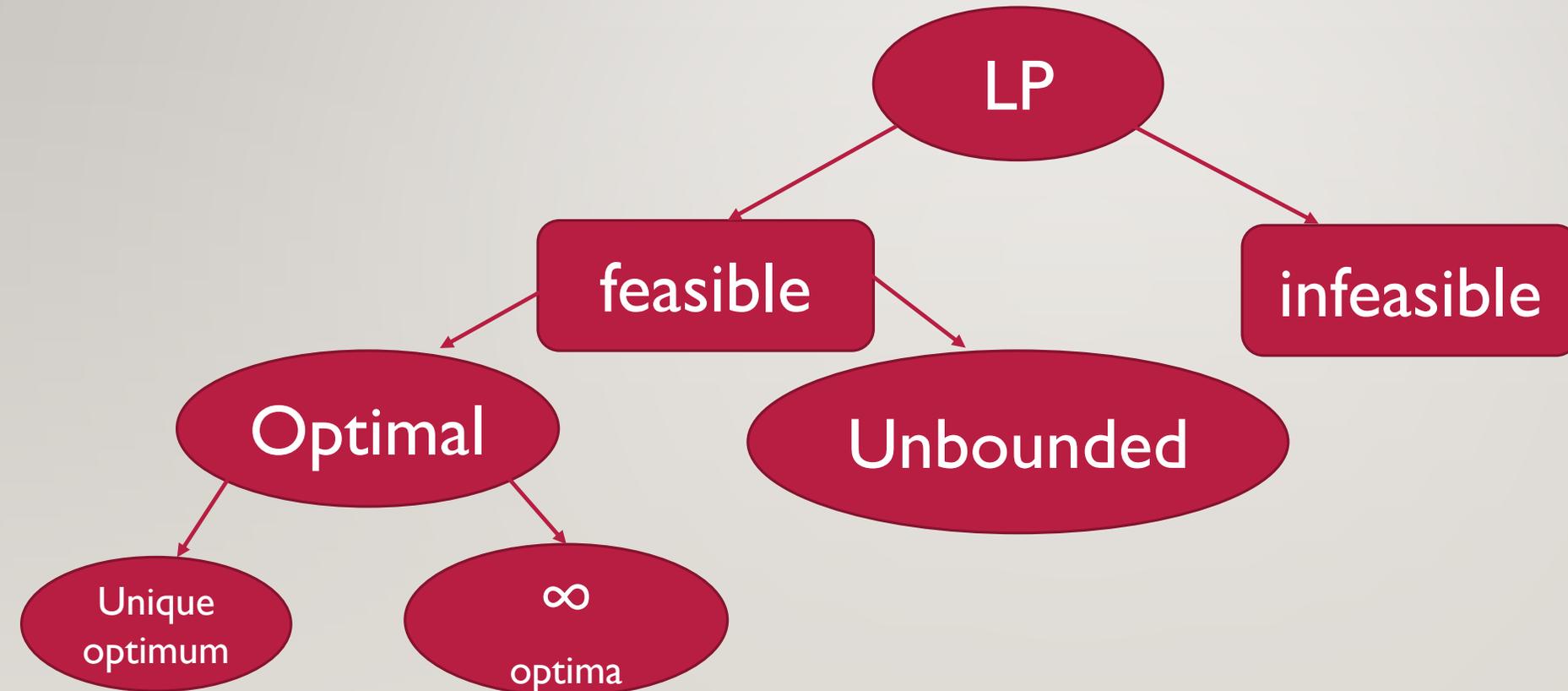
Interpretation: The store should spend \$6,666 on advertising and split this budget evenly between the magazine and online advertising.

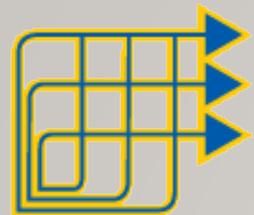
CLASSIFICATION OF LPs

An LP is called:

- **feasible** if it has at least one feasible solution and **infeasible** otherwise;
- **optimal** if it has an optimal solution;
- **unbounded** if it is feasible and its objective is not bounded (from above for a maximization problem and from below for a minimization problem) in the feasible region

CLASSIFICATION OF LPs





Thank you!

