V.M. GLUSHKOV INSTITUTE
OF CYBERNETICS OF THE NAS OF UKRAINE

LP (LINEAR PROGRAMMING) MODELING

LECTURE 4/SURVEY OF OPTIMIZATION

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AGENDA

- A Scheduling Problem
- Heavenly Pouch Inc. Problem revised
- Piecewise linear functions
- An Inventory Problem
- A Mixing Problem
- A transportation problem

A SCHEDULING PROBLEM

St. Tatiana Hospital uses a 12-hour shift schedule for its nurses, with each nurse working either day shifts (7:00 am–7:00 pm) or night shifts (7:00 pm–7:00 am). Each nurse works 3 consecutive day shifts or 3 consecutive night shifts and then has 4 days off. The hospital is aiming to design a schedule for day-shift nurses that minimizes the total number of nurses employed. The minimum number of nurses required for each day shift during a week is given in the following table:

A SCHEDULING PROBLEM (continued)

Day of week/shift	Nurses required
Monday (Mo)	16
Tuesday (Tu)	12
Wednesday (We)	18
Thursday (Th)	13
Friday (Fr)	15
Saturday (Sa)	9
Sunday (Su)	7

In addition, it is required that at least half of the day-shift nurses have weekends (Saturday and Sunday) off.

Formulate this problem as an LP.

A SCHEDULING PROBLEM (CONTINUED)

Step 1: Define the decision variables:

x_1 = the number of nurses working Mo-Tu-We schedule

x_2 = the number of nurses working Tu-We-Th schedule

x_3 = the number of nurses working We-Th-Fr schedule

x_4 = the number of nurses working Th-Fr-Sa schedule

x_5 = the number of nurses working Fr-Sa-Su schedule

x_6 = the number of nurses working Sa-Su-Mo schedule

x_7 = the number of nurses working Su-Mo-Tu schedule.

A SCHEDULING PROBLEM (continued)

Step 2: State the objective function:

The objective is to minimize the total number of nurses employed:

Minimize:

$$z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$$

A SCHEDULING PROBLEM (CONTINUED)

Step 3: Specify the *constraints*

$x_1 + x_6 + x_7 \geq 16.$ (Monday)

$x_1 + x_2 + x_7 \geq 12$ (Tuesday)

$x_1 + x_2 + x_3 \geq 18$ (Wednesday)

$x_2 + x_3 + x_4 \geq 13$ (Thursday)

$x_3 + x_4 + x_5 \geq 15$ (Friday)

$x_4 + x_5 + x_6 \geq 9$ (Saturday)

$x_5 + x_6 + x_7 \geq 7$ (Sunday).

Only the first three schedules do not involve working on weekends.

A SCHEDULING PROBLEM (continued)

Step 3: Specify the *constraints*

The requirement that *at least half* of the nurses have weekends off :

$$\frac{x_1 + x_2 + x_3}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7} \geq \frac{1}{2}$$

Convert into a standard form:

$$x_1 + x_2 + x_3 - x_4 - x_5 - x_6 - x_7 \geq 0.$$

A SCHEDULING PROBLEM (continued)

minimize $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$

subject to

$$
\begin{aligned}
x_1 \qquad\qquad\qquad + x_6 + x_7 &\geq 16 \\
x_1 + x_2 \qquad\qquad\qquad + x_7 &\geq 12 \\
x_1 + x_2 + x_3 \qquad\qquad\qquad &\geq 18 \\
x_2 + x_3 + x_4 \qquad\qquad &\geq 13 \\
x_3 + x_4 + x_5 \qquad &\geq 15 \\
x_4 + x_5 + x_6 &\geq 9 \\
x_5 + x_6 + x_7 &\geq 7 \\
x_1 + x_2 + x_3 - x_4 - x_5 - x_6 - x_7 &\geq 0 \\
x_1, x_2, x_3, x_4, x_5, x_6, x_7 &\geq 0.
\end{aligned}
$$

AMPL CODE FOR THE SCHEDULING PROBLEM

```
reset;
#Example Schedule for nurses prepared by Elena Volovyk;
# Decision variables;
var x1>=0, integer;  #the number of nurses working Mo-Tu-We schedule;
var x2>=0, integer;  #the number of nurses working Tu-We-Th schedule;
var x3>=0, integer;  #the number of nurses working We-Th-Fr schedule;
var x4>=0, integer;  #the number of nurses working Th-Fr-Sa schedule;
var x5>=0, integer;  #the number of nurses working Fr-Sa-Su schedule;
var x6>=0, integer;  #the number of nurses working Sa-Su-Mo schedule;
var x7>=0, integer;  #the number of nurses working Su-Mo-Tu schedule;

#objective function minimize the total number of nurses employed;
minimize staff: x1+x2+x3+x4+x5+x6+x7;
s.t. C1:x1+x6+x7>=16;  #first three consequitive days Mo-Tu-We;
s.t. C2:x1+x2+x7>=12;  #second three days;
s.t. C3:x1+x2+x3>=18;  #third three days;
s.t. C4:x2+x3+x4>=13;  #fourth three days;
s.t. C5:x3+x4+x5>=15;  #fifth three days;
s.t. C6:x4+x5+x6>=9;   #sixth three days;
s.t. C7:x5+x6+x7>=7;   #seventh three days;
option solver minos;
solve;
display staff,x1,x2,x3,x4,x5,x6,x7;
```

AMPL SOLUTION FOR THE SCHEDULING PROBLEM

E:\AMPL\ampl.mswin64\ampl.exe

```
ampl: reset;
ampl: #Example Scheduly for nurses prepared by ELena Volovk;
ampl: # Decision variables;
ampl: var x1>=0,integer;  #the number of nurses working Mo-Tu-We schedule;
ampl: var x2>=0, integer; #the number of nurses working Tu-We-Th schedule;
ampl: var x3>=0, integer; #the number of nurses working We-Th-Fr schedule;
ampl: var x4>=0, integer; #the number of nurses working Th-Fr-Sa schedule;
ampl: var x5>=0, integer; #the number of nurses working Fr-Sa-Su schedule;
ampl: var x6>=0, integer; #the number of nurses working Sa-Su-Mo schedule;
ampl: var x7>=0, integer;  #the number of nurses working Su-Mo-Tu schedule;
ampl:
ampl: #objective function minimize the total number of nurses employed;
ampl: minimize staff: x1+x2+x3+x4+x5+x6+x7;
ampl: s.t. C1:x1+x6+x7>=16;  #first three consequitive days Mo-Tu-We;
ampl: s.t. C2:x1+x2+x7>=12;  #second three days;
ampl: s.t. C3:x1+x2+x3>=18;  #third three days;
ampl: s.t. C4:x2+x3+x4>=13;  #fourth three days;
ampl: s.t. C5:x3+x4+x5>=15;  #fifth three days;
ampl: s.t. C6:x4+x5+x6>=9;   #sixth three days;
ampl: s.t. C7:x5+x6+x7>=7;  #seventh three days;
ampl: option solver minos;
ampl: solve;
MINOS 5.51: ignoring integrality of 7 variables
MINOS 5.51: optimal solution found.
2 iterations, objective 31
ampl: display staff,x1,x2,x3,x4,x5,x6,x7;
staff = 31
x1 = 11
x2 = 0
x3 = 10
x4 = 3
x5 = 2
x6 = 4
x7 = 1

ampl:
```

A SCHEDULING PROBLEM SOLUTION

This problem has multiple optimal solutions with z ∗ = 31. One of them is given by

$$x_1^* = 11, x_2^* = 0, x_3^* = 10, x_4^* = 3, x_5^* = 2, x_6^* = 4, x_7^* = 1,$$

Shift	# of nurses	Shift	# of nurses
Mo-Tu-We	11	Fr-Sa-Su	2
Tu-We-Th	0	Sa-Su-Mo	4
We-Th-Fr	10	Su-Mo-Tu	1
Th-Fr-Sa	3		

HEAVENLY POUCH Inc. PROBLEM REVISED

Heavenly Pouch, Inc. produces two types of baby carriers, non-reversible and reversible. Each non-reversible carrier sells for $23, requires 2 meters of a solid color fabric, and costs $8 to manufacture. Each reversible carrier sells for $35, requires 2 meters of a printed fabric as well as 2 meters of a solid color fabric, and costs $10 to manufacture. The company has 900 meters of solid color fabrics and 600 meters of printed fabrics available for its new carrier collection. It can spend up to $4,000 on manufacturing the carriers. The demand is such that all reversible carriers made are projected to sell, whereas at most 350 non-reversible carriers can be sold. Heavenly Pouch is interested in formulating a mathematical model that could be used to maximize its profit (e.g., the difference of revenues and expenses) resulting from manufacturing and selling the new carrier collection.

HEAVENLY POUCH INC. PROBLEM REVISED

ORIGINAL COMPLETE LINEAR PROGRAMMING FORMULATION

Maximize	$15x_1 + 25x_2$	(profit)
Subject to (s.t.)	$x_1 + x_2 \leq 450$	(solid color fabric constraint)
	$x_2 \leq 300$	(printed fabric constraint)
	$4x_1 + 5x_2 \leq 2000$	(budget constraint)
	$x_1 \leq 350$	(demand constraint)
	$x_1, x_2 \geq 0$	(nonnegativity constraints)

HEAVENLY POUCH INC. PROBLEM REVISED

Revised constraint

ORIGINAL	REVISED
$x_1 \leq 350$ (demand constraint)	the demand is such that at most 350 non-reversible carriers can be sold for a regular price of $23, but more carriers can be sold for a *discounted price* of $20 per carrier.

HEAVENLY POUCH INC. PROBLEM REVISED

Revised decision variable

x_1 = the number of non-reversible carriers sold for \$23,

x_1' = number of non-reversible carriers sold for \$20,

x_2 = the number of reversible carriers to manufacture.

HEAVENLY POUCH INC. PROBLEM REVISED

Revised LP-model

Maximize	$15x_1 + 12x_1' + 25x_2$	(profit)
Subject to (s.t.)	$x_1 + x_1' + x_2 \leq 450$	(solid color fabric constraint)
	$x_2 \leq 300$	(printed fabric constraint)
	$4x_1 + 4x_1' + 5x_2 \leq 2000$	(budget constraint)
	$x_1 \leq 350$	(demand constraint)
	$x_1, x_1', x_2 \geq 0$	(nonnegativity constraints)

AMPL CODE FOR HEAVENLY POUCH INC. PROBLEM REVISED

```
reset;
#Example Heavenly Pouch revised prepared by Elena Volovyk;
# Decision variables;
var x1>=0; #product 1 for $23;
var x3>=0; #product 1 for $20;
var x2>=0; #product 2;
#objective function maximize profit;
maximize profit: 15*x1+12*x3+25*x2;
s.t. C1:x1+x2+x3<=450;  #solid color fabric constraint;
s.t. C2:x2<=300;     #printed fabic constraint;
s.t. C3:4*x1+4*x3+5*x2<=2000;  #budget constraint;
s.t. C4:x1<=350;  #demand constraint;
option solver minos;
solve;
display profit,x1,x3,x2;
```

AMPL SOLUTION

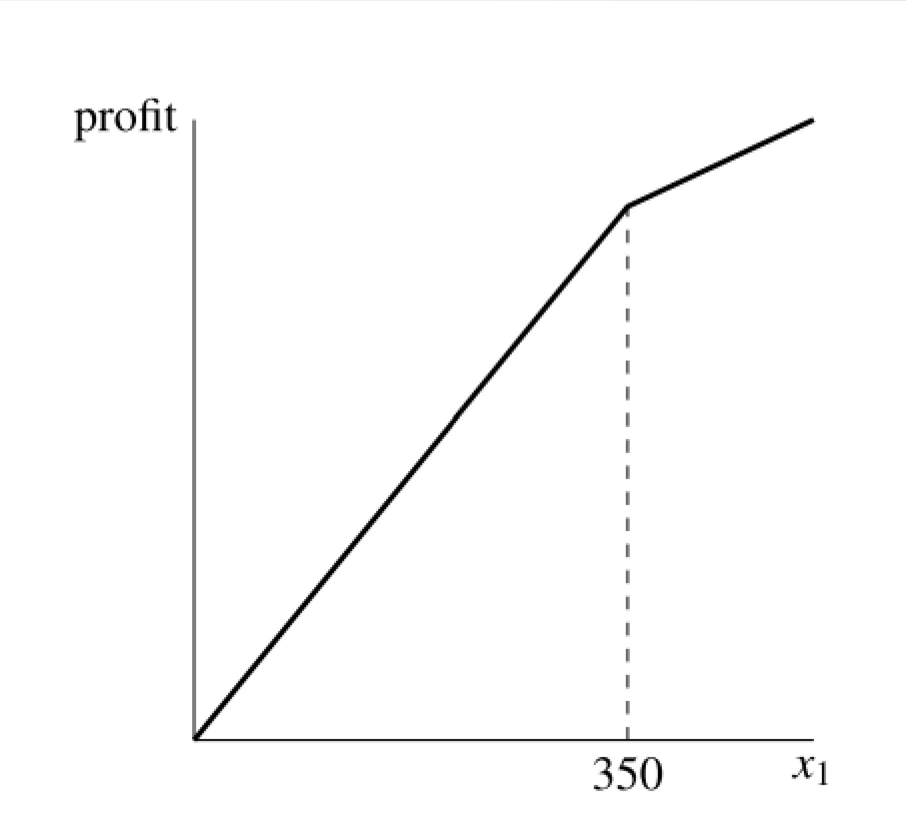
```
ampl: reset;
ampl: #Example Heavenly Pouch revised prepared by Elena Volovyk;
ampl: # Decision variables;
ampl: var x1>=0; #product 1 for $23;
ampl: var x3>=0; #product 1 for $20;
ampl: var x2>=0; #product 2;
ampl: #objective function maximize profit;
ampl: maximize profit: 15*x1+12*x3+25*x2;
ampl: s.t. C1:x1+x2+x3<=450;  #solid color fabric constraint;
ampl: s.t. C2:x2<=300;    #printed fabic constraint;
ampl: s.t. C3:4*x1+4*x3+5*x2<=2000;  #budget constraint;
ampl: s.t. C4:x1<=350;  #demand constraint;
ampl: option solver minos;
ampl: solve;
MINOS 5.51: optimal solution found.
2 iterations, objective 9375
ampl: display profit,x1,x3,x2;
profit = 9375
x1 = 125
x3 = 0
x2 = 300

ampl: _
```

PIECEWISE LINEAR FUNCTIONS

$$z_1 = \begin{cases} 15x_1, & \text{if } 0 \le x_1 \le 350, \\ 15 \times 350 + 12(x_1 - 350), & \text{if } x_1 > 350. \end{cases}$$

piecewise linear dependence

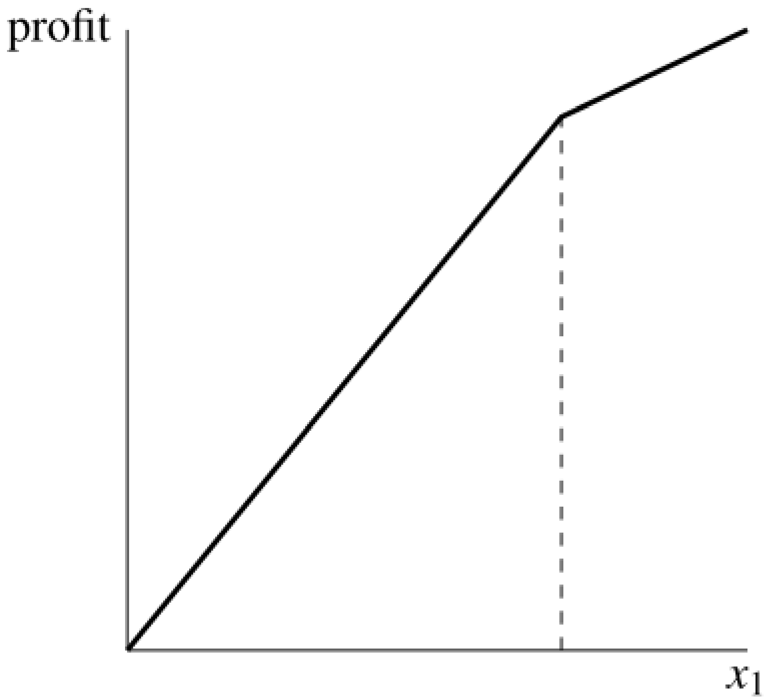


It is a function that is defined on an interval that is partitioned into segments such that over each segment, the function is linear

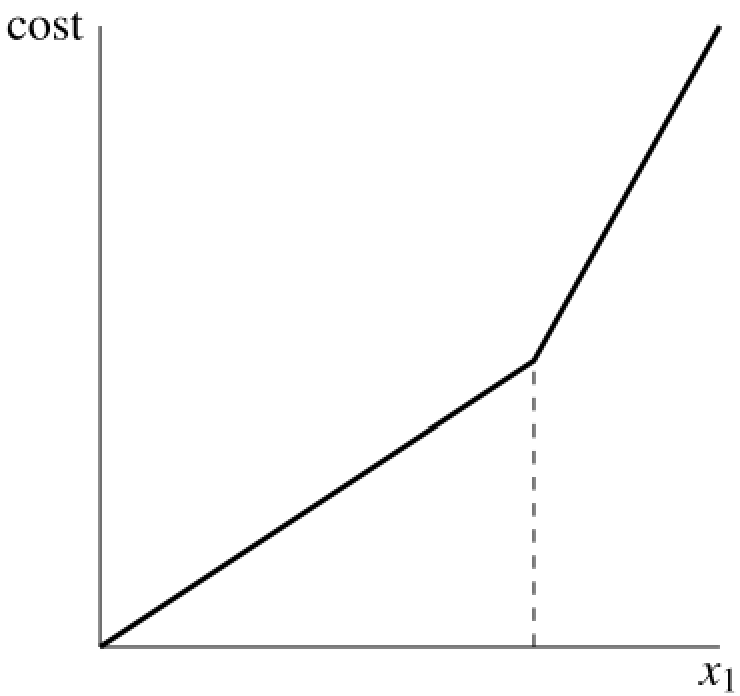
IMPORTANT REMARKS

- the slope in the piecewise linear function above decreases as x_1 increases above 350 (z_1 is a *concave* function of x_2),
- we are dealing with a maximization model,
- x_1' cannot be positive in an optimal solution, unless x_1 = 350.
- the coefficients for x_1 and x_1' differ in the objective, but are the same in each constraint,

CONCAVE MAXIMIZATION versus CONVEX MINIMIZATION



The slope is decreasing
Concave function
Suitable for maximization LP



The slope is increasing
Convex function
Suitable for minimization LP

AN INVENTORY MODEL

MaxJump LLC plans the monthly trampoline production quantities for the next quarter. The demand during the four months is

$$d_1 = 110,\ d_2 = 120,\ d_3 = 130,\ d_4 = 100$$

Presently, MaxJump has an inventory of 20 trampolines. During each month, MaxJump can manufacture up to 100 trampolines with regular-time labor for $120 per unit. With overtime labor, MaxJump can manufacture more trampolines, costing $150 per unit. A per unit inventory cost of $10 per unit is charged at the end of each month. The warehouse can fit up to 25 trampolines. MaxJump's management wants to develop a plan to minimize the total production and inventory costs for the quarter.

AN INVENTORY MODEL (cont.)

Index set: $T = \{ 1,2,3,4 \}$ - the planning horizon.

Decision variables:

x_t = number of units made using regular-time labor (\$120/unit) during month $t \in T$

y_t = number of units made using overtime labor (\$150/unit) during month $t \in T$

i_t = inventory level at the end of month $t \in T$ [inventory variables]

Note $i_t = i_{t-1} + (x_t + y_t) - d_t, \; t \in T,$ where $i_0 = 20$.

AN INVENTORY MODEL (cont.)

The total cost consists of three parts:

Regular-time production cost: $120\,(x_1 + x_2 + x_3 + x_4)$

Overtime production cost: $150\,(y_1 + y_2 + y_3 + y_4)$

Inventory cost: $10\,(i_1 + i_2 + i_3 + i_4)$

AN INVENTORY MODEL (final)

Minimize $z = 120\,(x_1 + x_2 + x_3 + x_4) + 150\,(y_1 + y_2 + y_3 + y_4) + 10\,(i_1 + i_2 + i_3 + i_4)$

subject to

$$x_t \leq 100, \quad t \in T$$

$$i_t = i_{t-1} + x_t + y_t - d_t, \quad t \in T$$

$$i_t \leq 25, \quad t \in T$$

$$x_t, y_t, i_t \geq 0, \quad t \in T$$

(where $i_0 = 20$)

AN INVENTORY MODEL (solution)

Objective = $54,100

x[*] = 100, 100, 100, 100

y[*] = 0, 10, 30, 0

i[*] = 10, 0, 0, 0

number of units made using regular-time labor ($120/unit) during month $t \in T$
number of units made using overtime labor ($150/unit) during month $t \in T$
inventory level at the end of month $t \in T$

DO IT YOURSELF ASSIGNMENT!!!

A MIXING PROBLEM

Painter Joe needs to complete a job that requires 50 gallons of brown paint and 50 gallons of gray paint. The required shades of brown and gray can be obtained by mixing the primary colors (red, yellow, and blue) in the proportions given in the following table.

Color	**Red**	**Yellow**	**Blue**
Brown	40%	30%	30%
Gray	30%	30%	40%

A MIXING PROBLEM

The same shades can be obtained by mixing secondary colors (orange, green, and purple), each of which is based on mixing two out of three primary colors in equal proportions (red/yellow for orange, yellow/blue for green, and red/blue for purple). Joe currently has 20 gallons each of red, yellow, and blue paint, and 10 gallons each of orange, green, and purple paint. If needed, he can purchase any of the primary color paints for $20 per gallon, however he would like to save by utilizing the existing paint supplies as much as possible.

Formulate an LP helping Joe to minimize his costs.

A MIXING PROBLEM

- (source colors) red, yellow, blue, orange, green, purple - 6 colors (index i)
- (target colors) brown and gray - 2 colors (index j)

Decision variables

x_{ij} = gallons of paint of color i used to obtain color j paint for $i \in \{ 1,...,6 \}$, $j \in \{ 1,2 \}$

x_i = gallons of paint i purchased, i = 1,2,3.

Availability: 20 gal of red, yellow, blue
10 gal of orange, green, purple

Requirement: at least 50 gal of brown
at least 50 gal of gray

A MIXING PROBLEM

Objective is to minimize $z = 20x_1 + 20x_2 + 20x_3$

Specify the constraints:

- The total amount of brown and gray paint made must be at least 50 gallons each:

$$x_{11} + x_{21} + x_{31} + x_{41} + x_{51} + x_{61} \geq 50$$

$$x_{12} + x_{22} + x_{32} + x_{42} + x_{52} + x_{62} \geq 50$$

A MIXING PROBLEM

Specify the constraints:

- The amount of each paint used for mixing must not exceed its availability:

$$x_{11} + x_{12} \leq 20 + x_1$$
$$x_{21} + x_{22} \leq 20 + x_2$$
$$x_{31} + x_{32} \leq 20 + x_3$$
$$x_{41} + x_{42} \leq 10$$
$$x_{51} + x_{52} \leq 10$$
$$x_{61} + x_{62} \leq 10.$$

A MIXING PROBLEM

Specify the constraints:

- three out of six colors used for mixing contain red, and the total amount of red paint (including that coming from orange and purple paints) used in the brown mix is:

$$x_{11} + 0.5x_{41} + 0.5x_{61}$$

A constraint for the proportion of red color in the brown mix

$$\frac{x_{11} + 0.5x_{41} + 0.5x_{61}}{x_{11} + x_{21} + x_{31} + x_{41} + x_{51} + x_{61}} = 0.4$$

A MIXING PROBLEM

Specify the constraints:

- three out of six colors used for mixing contain red, and the total amount of red paint (including that coming from orange and purple paints) used in the brown mix is:

$$x_{11} + 0.5x_{41} + 0.5x_{61}$$

A constraint for the proportion of red color in the brown mix

$$\frac{x_{11} + 0.5x_{41} + 0.5x_{61}}{x_{11} + x_{21} + x_{31} + x_{41} + x_{51} + x_{61}} = 0.4$$

A MIXING PROBLEM

Specify the constraints:

- three out of six colors used for mixing contain red, and the total amount of red paint (including that coming from orange and purple paints) used in the brown mix is:

$$x_{11} + 0.5x_{41} + 0.5x_{61}$$

A constraint for the proportion of red color in the brown mix

$$\frac{x_{11} + 0.5x_{41} + 0.5x_{61}}{x_{11} + x_{21} + x_{31} + x_{41} + x_{51} + x_{61}} = 0.4$$

Convert to linear →

→ $$0.6x_{11} - 0.4x_{21} - 0.4x_{31} + 0.1x_{41} - 0.4x_{15} + 0.1x_{61} = 0$$

A MIXING PROBLEM

The proportion of yellow and blue colors in the brown mix:

$$\frac{x_{21} + 0.5x_{41} + 0.5x_{51}}{x_{11} + x_{21} + x_{31} + x_{41} + x_{51} + x_{61}} = 0.3 \quad \text{yellow}$$

Convert to linear

$$\frac{x_{31} + 0.5x_{51} + 0.5x_{61}}{x_{11} + x_{21} + x_{31} + x_{41} + x_{51} + x_{61}} = 0.3 \quad \text{blue}$$

$$-0.3x_{11} + 0.7x_{21} - 0.3x_{31} + 0.2x_{41} + 0.2x_{51} - 0.3x_{61} = 0$$

$$-0.3x_{11} - 0.3x_{21} + 0.7x_{31} - 0.3x_{41} + 0.2x_{51} + 0.2x_{61} = 0$$

A MIXING PROBLEM

the proportion of each of the primary colors in the gray paint mix:

$$0.7x_{12} - 0.3x_{22} - 0.3x_{32} + 0.2x_{42} - 0.3x_{52} + 0.2x_{62} = 0 \qquad \text{(red)}$$

$$-0.3x_{12} + 0.7x_{22} - 0.3x_{32} + 0.2x_{42} + 0.2x_{52} - 0.3x_{62} = 0 \qquad \text{(yellow)}$$

$$-0.4x_{12} - 0.4x_{22} + 0.6x_{32} - 0.4x_{42} + 0.1x_{52} + 0.1x_{62} = 0 \qquad \text{(blue)}$$

A MIXING PROBLEM (final)

Minimize $20x_1 +20x_2 +20x_3$

Subject to

$$0.6x_{11} - 0.4x_{21} - 0.4x_{31} + 0.1x_{41} - 0.4x_{15} +0.1x_{61} = 0$$

$$-0.3x_{11} + 0.7x_{21} - 0.3x_{31} + 0.2x_{41} + 0.2x_{51} - 0.3x_{61} = 0$$

$$-0.3x_{11} - 0.3x_{21} + 0.7x_{31} - 0.3x_{41} + 0.2x_{51} + 0.2x_{61} = 0$$

$$0.7x_{12} - 0.3x_{22} - 0.3x_{32} + 0.2x_{42} - 0.3x_{52} + 0.2x_{62} = 0$$

$$-0.3x_{12} + 0.7x_{22} - 0.3x_{32} + 0.2x_{42} + 0.2x_{52} - 0.3x_{62} = 0$$

$$-0.4x_{12} - 0.4x_{22} +0.6x_{32} - 0.4x_{42} + 0.1x_{52} + 0.1x_{62} = 0$$

$$x_{11} +x_{21} +x_{31} +x_{41} +x_{51} +x_{61} \geq 50$$

$$x_{12} +x_{22} +x_{32} +x_{42} +x_{52} +x_{62} \geq 50$$

A MIXING PROBLEM (final)

Minimize $20x_1 + 20x_2 + 20x_3$

Subject to

$$x_{11} + x_{12} - x_1 \le 20$$
$$x_{21} + x_{22} - x_2 \le 20$$
$$x_{31} + x_{32} - x_3 \le 20$$

(Availability constraints)

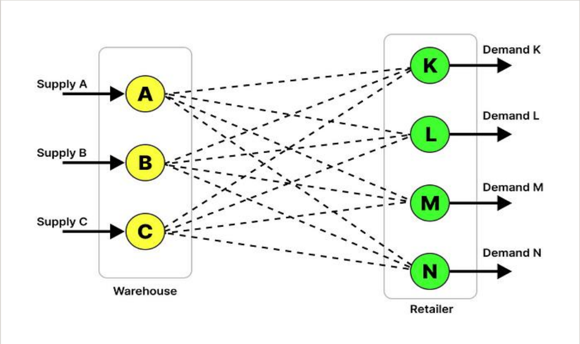
$$x_{41} + x_{42} \le 10$$

$$x_{51} + x_{52} \le 10$$

$$x_{61} + x_{62} \le 10.$$

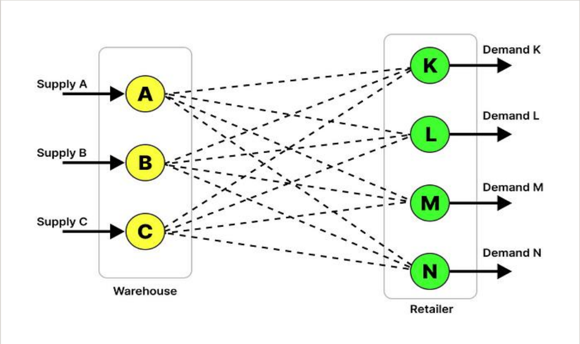
$$x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}, x_{41}, x_{42}, x_{51}, x_{52}, x_{61}, x_{62}, x_1, x_2, x_3 \ge 0.$$

A TRANSPORTATION PROBLEM



A wholesale company specializing in one product has m warehouses W_i , $i = 1,...,m$ serving n retail locations R_j , $j = 1,...,n$. Transporting one unit of the product from W_i to R_j costs c_{ij} dollars, $i = 1,...,m$, $j = 1,...,n$. The company has s_i units of product available to ship from W_i , $i = 1,...,m$. To satisfy the demand, at least d_j units of the product must be delivered to R_j .

Formulate an LP to decide how many units of the product should be shipped from each warehouse to each retail location so that the company's overall transportation costs are minimized.



A TRANSPORTATION PROBLEM

The decision variables are

x_{ij} = the product quantity shipped from Wi to R_j , i = 1,...,m; j = 1,...,n.

All variables must be nonnegative.

The objective is to minimize the total cost of transportation:

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Make sure that the number of units shipped out of W_i does not exceed s_i :

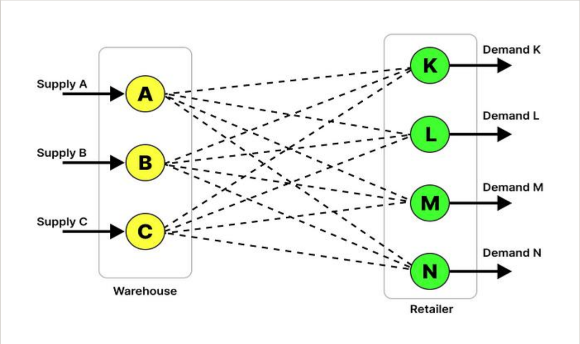
$$\sum_{j=1}^{n} x_{ij} \leq s_i$$

where I = 1,…., m

$$\sum_{i=1}^{m} x_{ij} \geq dj$$

Satisfy the demand at Rj

j=1,…, n



A TRANSPORTATION PROBLEM (final)

minimize

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} \leq s_i \text{ , i = 1,...., m}$$

$$\sum_{i=1}^{m} x_{ij} \geq dj \text{ . j = 1,..., n}$$

$x_{ij} \geq 0$, i = 1,..., m, j = 1,..., n

Thank you!