

V.M. GLUSHKOV INSTITUTE  
OF CYBERNETICS OF THE NAS OF UKRAINE

# MATHEMATICAL PROGRAMMING BASICS

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LECTURE 2/SURVEY OF OPTIMIZATION

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# AGENDA

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- Mathematical Programming Applications
- Definition of Mathematical Programming
- Mathematical Programming Branches
- Formulating of Optimization Problems
- Terminology and Definitions (decision variables, objective function, feasible region, binding/nonbinding constraint, local/global extremum, etc.)
- Formulating a Linear Programming Model
- Optimization Problem Examples

# MATHEMATICAL PROGRAMMING APPLICATIONS

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Mathematical programming is a branch of operations research, which comprises a wide class of control problems the mathematical models of which are finite-dimensional extremum problems.

It finds applications in various areas of human activity where it is necessary to choose one of the possible ways of action, e.g. in solving numerous problems of control and planning of production processes as well as in problems of design and long-term planning.

# DEFINITION OF MATHEMATICAL PROGRAMMING

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**Mathematical programming** is theoretical tool of management science and economics in which management operations are described by mathematical equations that can be manipulated for a variety of purposes. If the basic descriptions involved take the form of linear algebraic equations, the technique is described as linear programming. If more complex forms are required, the term nonlinear programming is applied. (Britannica, The Editors of Encyclopaedia. "mathematical programming". *Encyclopedia Britannica*, <https://www.britannica.com/science/mathematical-programming>. Accessed 1 April 2023.)

# MATHEMATICAL PROGRAMMING BRANCHES

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In mathematical programming the following branches are distinguished:

- **Linear programming:** The objective function  $\phi(x)$  and the constraints  $q_i(x)$  and  $h_j(x)$  are linear.
- **Quadratic programming:** The objective function is quadratic and convex, while the feasible set is defined by linear equalities and inequalities.
- **Convex programming:** The objective function and the constraints are convex.
- **Discrete programming/Integer programming:** The solution is sought only at discrete, say integer, points of  $X$ .

# MATHEMATICAL PROGRAMMING BRANCHES

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- **Stochastic programming:** In contrast to deterministic problems, the data contain an element of indeterminacy.

For example: in stochastic problems of minimization of a linear function

$$\sum_{j=1}^n c_j x_j$$

under linear constraints

$$\sum_{j=1}^n a_{ij} x_j \geq b_i, i = 1, 2, \dots$$

the parameters  $c_j$ ,  $a_{ij}$ ,  $b_i$ , or only some of them, are random.

# FORMULATING AN OPTIMIZATION PROBLEM

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- **Step 1**: introduce the *decision variables* - the parameters whose values need to be determined in order to solve the problem;
- **Step 2**: state the *objective function* – the quantity is to be optimized as a function of the decision variables;
- **Step 3**: specify the *constraints* – the conditions that the design parameters are required to satisfy.

# FORMULATING AN OPTIMIZATION PROBLEM

## *EXAMPLE 1*

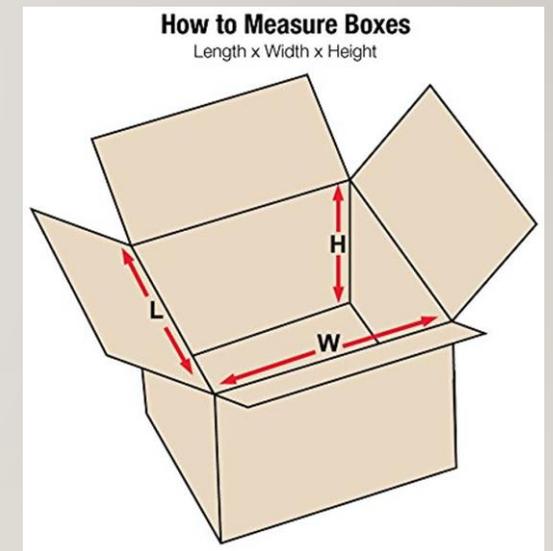
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Create a box with an open top and square base with a surface area of 300 square centimeters .What height will produce a box with the maximum possible volume?

**Step 1:** the decision variables : h – the height

x – the side measurement

**Step 2:** state the objective function:  $V = x^2 h \longrightarrow$  maximum



# FORMULATING AN OPTIMIZATION PROBLEM

## *EXAMPLE 1*

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Create a box with an open top and square base with a surface area of 300 square centimeters .What height will produce a box with the maximum possible volume?

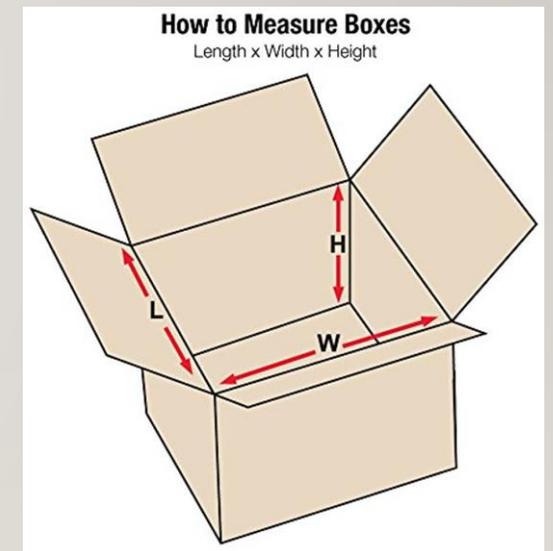
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**Step 3:** specify the constraints

The surface area of the box is a constant of 300 square centimeters.

This is the constraint. Turn this into an equation of the surface area:

$$SA = x^2 + 4xh = 300$$



# FORMULATING AN OPTIMIZATION PROBLEM

## *EXAMPLE 1*

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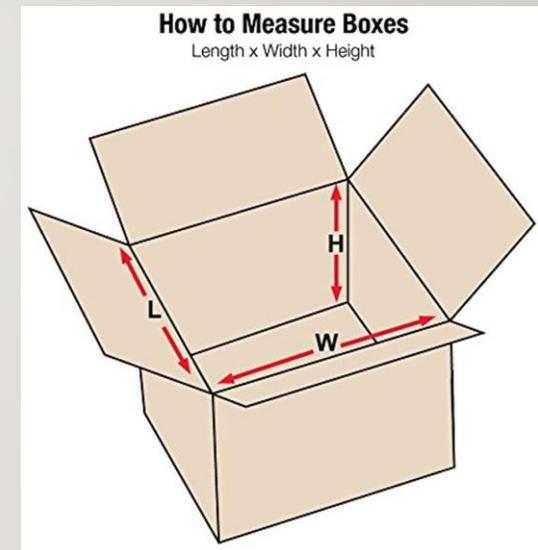
In summary the optimization problem is:

MAXIMIZE  $V = x^2 h$

Subject to  $x^2 + 4xh = 300$

$x > 0$

$h > 0$



# FORMULATING AN OPTIMIZATION PROBLEM

## TERMINOLOGY

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Let  $X \subseteq \mathbb{R}^n$  be a set of n-dimensional real vector in the form of

$$x = [x_1, x_2, \dots, x_n]^T$$

and

$f: X \rightarrow \mathbb{R}$  a given function .

Optimization general form:

Maximize  $f(x)$   
Subject to  $x \in X$

or

Minimize  $f(x)$   
Subject to  $x \in X$

$\max f(x)$   
 $x \in X$

$\min f(x)$   
 $x \in X$

# FORMULATING AN OPTIMIZATION PROBLEM

## TERMINOLOGY

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where  $x_j, j = 1, \dots, n$  – is called a *decision variable*,  $X$  is called the *feasible region*,  
 $f$  is the *objective function*,

Conversion :

$$\max_{x \in X} f(x) = \min_{x \in X} (-f(x))$$

Set of constraints:

$$\begin{aligned} h_i(x) &= 0, i \in E \\ g_j(x) &\leq 0, j \in \Phi \end{aligned} \quad \text{where} \quad \begin{aligned} h_i: \mathbb{R}^n &\rightarrow \mathbb{R}, i \in E \\ g_j: \mathbb{R}^n &\rightarrow \mathbb{R}, j \in \Phi \end{aligned}$$

$E$  and  $\Phi$  are index sets

# FORMULATING AN OPTIMIZATION PROBLEM

## POSSIBLE SCENARIOS

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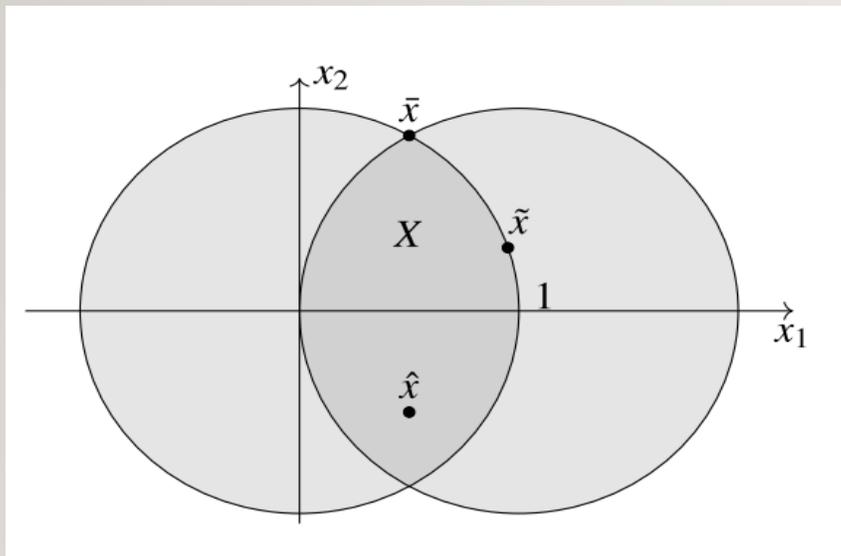
- decision variables are *binary*:  $x_j \in \{0, 1\} \iff x_j(1 - x_j) = 0$
- *inequality* constraints:  $g_1(x) \geq g_2(x) \iff g_3(x) \leq 0$ , where  $g_3(x) = g_2(x) - g_1(x)$   
 $g_1(x) \geq g_2(x) \iff g_2(x) - g_1(x) \leq 0$
- If the feasible region  $X$  is described by a set of equality and inequality constraints,  
 $X = \{x \in \mathbb{R}^n : h_i(x) = 0, g_j(x) \leq 0, i \in E, j \in \Phi\}$   
a point  $\bar{x}$  is *feasible* if and only if it satisfies each of the equality and inequality constraints defining the feasible region  $X$ .

# BINDING CONSTRAINT DEFINITION

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An inequality constraint  $g_i(x) \leq 0$  is called *binding* or *active* at a feasible point  $\bar{x}$  if  $g_i(\bar{x}) = 0$ . Otherwise, if  $g_i(\bar{x}) < 0$ , the constraint is called nonbinding or inactive at  $\bar{x}$ .

Example:



Consider the feasible set  $X$  defined by two inequality constraints,

$$X = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1, (x_1 - 1)^2 + x_2^2 \leq 1\}.$$

$\hat{x}$  — none is active/binding

$\tilde{x}$  — the first constraint is active/binding

$\bar{x}$  — both are active/binding

# MORE DEFINITIONS

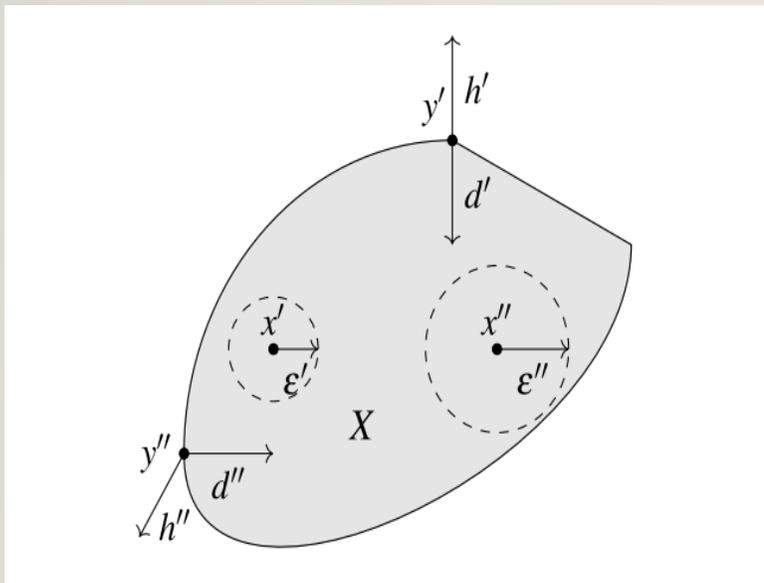
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- A vector  $d \in \mathbb{R}^n$  is called a *feasible direction* for set  $X \subseteq \mathbb{R}^n$  at  $\bar{x} \in X$  if there exists  $\delta > 0$  such that  $\bar{x} + \alpha d \in X$  for any  $\alpha \leq \delta$ .
- $\hat{x} \in X$  is an *interior point* of  $X$  if there exists  $\varepsilon > 0$ , such that  $\hat{x}$  is included in  $X$  together with the  $\varepsilon$ -ball centered at  $\hat{x}$ , i.e.,  $B(\hat{x}, \varepsilon) \subset X$ . We call the set of all interior points of  $X$  *the interior of  $X$* , denoted by  $\text{int}(X)$ .

# MORE DEFINITIONS

- $\bar{x} \in X$  is a *boundary point* of  $X$  if it is not an interior point of  $X$ .
- Any direction is feasible at an interior point.

Example:



$x'$  and  $x''$  are interior points

$y'$  and  $y''$  are boundary points

$d'$  and  $d''$  are feasible directions

$h'$  and  $h''$  are infeasible directions

# MORE DEFINITIONS

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- The point  $x^* \in X$  is a point of **global minimum** (global minimizer) for the problem  $\min_{x \in X} f(x)$  (denoted as  $x^* = \operatorname{argmin}_{x \in X} f(x)$ ) if  $f(x^*) \leq f(x)$  for all  $x \in X$ .

- A global minimizer does not always exist, even if the function is bounded from below.
- For a function  $f: X \rightarrow \mathbb{R}$ , its greatest lower bound is called the **infimum** and is denoted by

$$\inf_{x \in X} f(x)$$

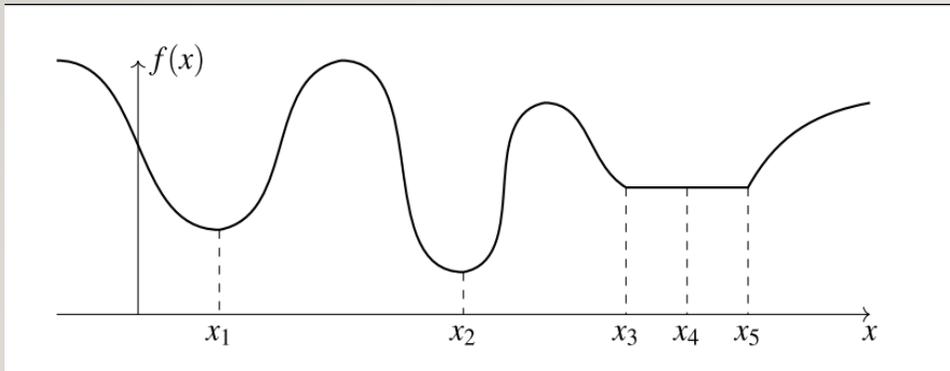
and its least upper bound is called the **supremum** and is denoted by  $\sup_{x \in X} f(x)$

Example :  $(a,b)$  . then  $\inf(a,b)=a$  and  $\sup(a,b) = b$

# MORE DEFINITIONS

- If  $f$  is unbounded from below or above,  $\inf_{x \in X} f(x) = -\infty$  or  $\sup_{x \in X} f(x) = \infty$  respectively.
- The point  $x^* \in X$  is a point of **local minimum** (local minimizer) for the problem  $\min_{x \in X} f(x)$  if there exists  $\varepsilon > 0$  such that

$$f(x) \geq f(x^*) \text{ for any } x \in X \text{ with } \|x - x^*\| \leq \varepsilon.$$



# LINEAR FUNCTION DEFINITION

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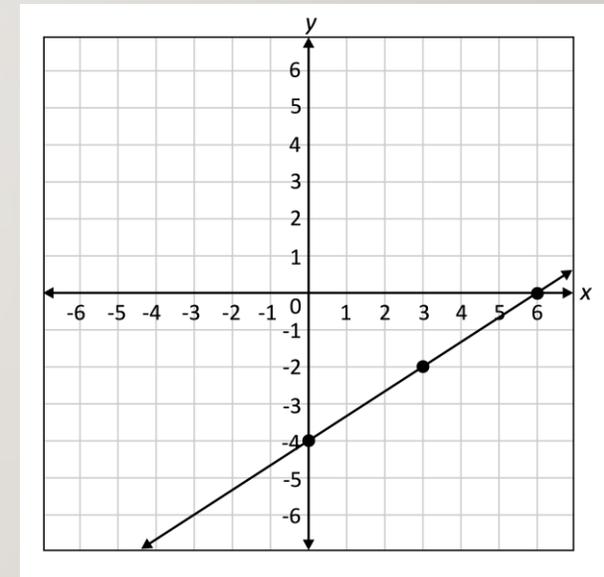
A function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  in the form:

$$f(x) = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

where

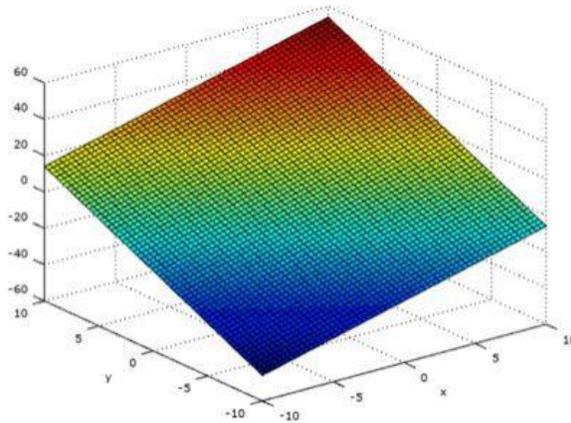
- $c_j, j= 1, \dots, n$  are constants
- $x_j, j= 1, \dots, n$  are variables

is called **linear**.

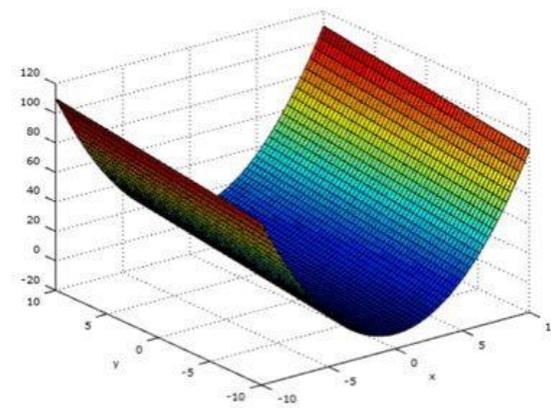


# LINEAR FUNCTION DEFINITION

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$f(x) = 2x - 3y + 4$   
Linear function



$f(x) = x^2 + y$   
Non-linear function

# FORMULATING A LINEAR PROGRAMMING MODEL

## EXAMPLE

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Heavenly Pouch, Inc. produces two types of baby carriers, non-reversible and reversible. Each non-reversible carrier sells for \$23, requires 2 meters of a solid color fabric, and costs \$8 to manufacture. Each reversible carrier sells for \$35, requires 2 meters of a printed fabric as well as 2 meters of a solid color fabric, and costs \$10 to manufacture. The company has 900 meters of solid color fabrics and 600 meters of printed fabrics available for its new carrier collection. It can spend up to \$4,000 on manufacturing the carriers. The demand is such that all reversible carriers made are projected to sell, whereas at most 350 non-reversible carriers can be sold. Heavenly Pouch is interested in formulating a mathematical model that could be used to maximize its profit (e.g., the difference of revenues and expenses) resulting from manufacturing and selling the new carrier collection.

# FORMULATING A LINEAR PROGRAMMING MODEL EXAMPLE

	Non-reversible carrier	Reversible carrier	Limit
Selling Price	\$23	\$35	
Solid fabric	2	2	900
Printed fabric	0	2	600
Manufacture cost	\$8	\$10	4,000
Marketing Constraint	$\leq 350$		

# FORMULATING A LINEAR PROGRAMMING MODEL EXAMPLE

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- **Step 1**: introduce the *decision variables* - the parameters whose values need to be determined in order to solve the problem;

$x_1$  = the number of non-reversible carriers to manufacture

$x_2$  = the number of reversible carriers to manufacture.

# FORMULATING A LINEAR PROGRAMMING MODEL EXAMPLE

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- **Step 2:** state the *objective function* – the quantity is to be optimized as a function of the decision variables;

The objective of Heavenly Pouch, Inc. is to **maximize** its profit, which is the difference of the revenues and manufacturing costs.

$$\text{Revenue } r = 23x_1 + 35x_2$$

$$\text{Cost } c = 8x_1 + 10x_2$$

$$\text{Profit } z = r - c = (23x_1 + 35x_2) - (8x_1 + 10x_2) = 15x_1 + 25x_2$$

# FORMULATING A LINEAR PROGRAMMING MODEL EXAMPLE

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- **Step 3:** specify the *constraints* – the conditions that the design parameters are required to satisfy.

1. the company has only 900 meters of solid color fabrics available,

$$2x_1 + 2x_2 \leq 900 \quad \leftrightarrow \quad x_1 + x_2 \leq 450$$

2. at most 600 meters of printed fabrics can be used,

$$2x_2 \leq 600 \quad \leftrightarrow \quad x_2 \leq 300$$

# FORMULATING A LINEAR PROGRAMMING MODEL EXAMPLE

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- **Step 3:** specify the *constraints* – the conditions that the design parameters are required to satisfy.

3. Heavenly Pouch, Inc. spends \$8/carrier and \$10/carrier to manufacture a non-reversible and reversible carrier respectively, and it can spend up to \$4,000 on manufacturing the carriers

$$8x_1 + 10x_2 \leq 4000 \quad \leftrightarrow \quad 4x_1 + 5x_2 \leq 2000$$

# FORMULATING A LINEAR PROGRAMMING MODEL EXAMPLE

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- **Step 3**: specify the *constraints* – the conditions that the design parameters are required to satisfy.

4. At most 350 non-reversible carriers can be sold

$$x_1 \leq 350$$

# THE COMPLETE LINEAR PROGRAMMING FORMULATION

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Maximize  $15x_1 + 25x_2$  (profit)

Subject to (s.t.)  $x_1 + x_2 \leq 450$  (solid color fabric constraint)

$x_2 \leq 300$  (printed fabric constraint)

$4x_1 + 5x_2 \leq 2000$  (budget constraint)

$x_1 \leq 350$  (demand constraint)

$x_1, x_2 \geq 0$  (nonnegativity constraints)

# HINTS FOR FORMULATING *LP* OPTIMIZATION MODEL

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1. **Decision variables:** if you are using indices (e.g.,  $x_i$ ), clearly specify the meaning of indices and their range of values (e.g.,  $i$  - index for resources available,  $i = 1, \dots, n$ ).
2. **Objective function:** when writing the objective function, remember to indicate whether you are minimizing (min) or maximizing (max).
3. **Constraints:** make sure that you specify whether each variable is nonnegative or real.

# Thank you!

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