Approaches for solving informational extended games

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1. Introduction

Game theory and informational aspects. Information issues are very important for mathematical modeling decisions makig problems in situations of risk and conflict.

Last years the informational aspect represents a real stimulus for the elaboration of the new study methods for non-cooperative game theory. The informational aspect in the game theory is manifested by: the information about strategy's choice, the payoff functions, the order of moves, and optimal principles of players; the using methods of possessed information in the strategy's choice by players. The problems with access to information generate conditions of *perfect*, *imperfect*, *complete*, *incomplete*, uncertain or asymmetric information. And so, inclusions of information as an important element of game have imposed a new structure to the game theory: the games in complete information (the games in extended form), the games in not complete information and the games in imperfect information (the Bayes games). The player's possession of supplementary information about unfolding of the game can influence appreciably the player's gains. This can be observed as follows from: the role of knowledge of information by the first player about the strategy's choice by the second player, the role of the information about the optimal principles used by players. For the mathematical modeling of the decision-making problems in situations of risk, uncertainty and informational impact a new class of games is studied. Different ways of solving games in complete and perfect information (or over the set of informational extended strategies) are studied. In the paragraphs below is a theoretical and practical analysis of how to solve games with informational extended strategies.

The Nash and Bayes-Nash solutions for informational extended games are discussed. We propose to use the new methodology for solving the complete and perfect information game. To solve the games of these type we construct the incomplete and imperfect information game generated by the informational extended strategies. Then we construct associated Bayesian game with non-informational extended strategies and determine Bayes-Nash equilibria in the incomplete information games.

In the given presentation we will focus on the following:

- Normal forms of the one and double–way informational extended game.
- Properties of the Nash Equilibrium Profiles in the Informational Extended Games.
- Informational non-Extended Game Generated by the Informational Extended Strategies.
    - ✓ Informational extended strategies in the duopol games.
- Converting the Two Persons Game with Informational Extended Strategies to Bayesian Game.
- Approaches for solving bimatrix informational extended games.
- Find the Bayes-Nash solutions in the bimatrix informational extended games using bayesian subgames.

2. Normal forms of the informational extended game

2.1 Normal forms of the one–way informational extended game

One of the main problem in the game theory is to define (or to construct) the normal form (or strategic form) of the game using the "verbal" description of the game. Normal form allows to begin the more detailed studies of the all approaches to solve the game. The strategic form or normal form of the game consists of a number of players, a set of strategies for each of the players, and a payoff function that associates a payoff to each player with a choice of strategies by each player.

Let

$$\Gamma = \langle \mathrm{I}; \mathrm{X_i}, \mathrm{i} \in \mathrm{I}; \mathrm{H_i}: \mathrm{X} \to \mathbb{R} \rangle$$

be the strategic form or normal form of the static games, where $I = \{1,2,\dots,n\}$ is the set of the players, X_i is a set of available alternative of the player $i \in I$, H_i: $X_i \to R$ is the payoff function of the player $i \in I$ and $X = \prod_{i\in I} X_i$ is the set of strategy profiles for the game.

We will define the informational extension of the game Γ, generated by a one-way directional informational flow (Novac 2008), denoted by "$j \to i$", which meaning is: the player i, and only him, knows exactly what value of the strategy will be choose by the player j. In the shot we named this games as *"one–way $j \to i$ informational extended game"* or *"$j \to i$ games"*. In this case the game is done in the following way: the player i independently will chose the "actions program" as an respond to the non extended informational strategies chose by the player j. The strategy of the player i will be defined by taking in account of all accessible informations about the choose strategy of the player j. Namely, the informational extended strategy will be constructed by the following deduction: "if the strategy x_j of the player j belongs to the specified domain then the player i will choose the specified strategy x_i" etc. In the other hand, the informational distended strategy of the player i is a response function of the player i from the chosen strategy of the player j.

The normal form of the one-way $j \to i$ informational extended game will be

$$\Gamma(\mathrm{j} \to \mathrm{i}) = \left\langle \mathrm{I}, \Theta_\mathrm{i}, \mathrm{X}_{[-\mathrm{i}]}, \left\{\widetilde{\mathrm{H}}_\mathrm{p}\right\}_{\mathrm{p}\in\mathrm{I}} \right\rangle,$$

where the set of the informational extended strategies of the player i is $\Theta_\mathrm{i} = \left\{\theta_i^k: X_j \to X_i^k, k = 1,\dots,\chi_i\right\}$, $X_i^k \subseteq X_i$ is the action set strategy for the player i, for all informational extended strategy $\theta_\mathrm{i}^\mathrm{k}$ define the set $\mathrm{X}\left(\theta_\mathrm{i}^\mathrm{k}\right) = \left\{\left(\mathrm{x}_\mathrm{i}^\mathrm{k}, x_j, \mathrm{x}_{-[\mathrm{i,j}]}\right): \mathrm{x}_\mathrm{i}^\mathrm{k} = \theta_\mathrm{i}^\mathrm{k}\left(x_j\right), \forall x_j \in \mathrm{X_j}, \forall \mathrm{x}_{-[\mathrm{i,j}]} \in \mathrm{X}_{[-\mathrm{ij}]}\right\}$ of the strategy profiles for the non informational extended game Γ generated by the θ_i^k strategy and the payoff functions[1]

$$\widetilde{H}_p(\theta_i^k, x_j, x_{-[i,j]}) = H_p(x_i^k, x_j, x_{-[i,j]}),\ (x_i^k, x_j, x_{-[i,j]}) \in X(\theta_i^k).$$

2.2 Normal forms of the double-way informational extended game

Now we construct the normal form of the informational extension of the game generated by a double-way informational flow (Hancu 2011), denoted by $i \leftrightarrow j$.

For these games we assume that, from the informational point of view, the player *i* is of the " $j \to i$ informational type" and simultaneously, the player *j* is of the " $i \to j$ informational type". It means that at any time the player *i* knows exactly the value of the strategy chosen by the player *j*, as well as, simultaneously, the player *j* knows exactly the value of the strategy chosen by the player *i*. It should be mentioned that the players *don't know the informational type of each other*. In other words the players don't known the informational extended strategies of each others and from this point of view we can consider that game is in *incomplete information* structure[2]. In this case the game could be done in the following way: the player *i* independently will chose an "actions program" as a respond to the non extended informational strategies chosen by the player *j* and simultaneously the player *j* independently will chose an "actions program" as a respond to the non extended informational strategies chosen by the player *i*.

The action sets strategy of the payer *i* **(**correspondingly of the player *j*) noted by X_i^α (correspondingly X_j^β), represent the sets of the non informational extended strategy chosen by the player *i* (correspondingly by the player *j*) in the capacity of the response to the every chosen strategy by the player *j* (correspondingly by the player *i*).

As mention above, informational extended strategy of the player is the detailed agenda of the chosen the value of the strategies from the action sets strategy. These agenda can be defined using the functions of the following type $\theta_i^\alpha: X_j \to X_i^\alpha$ that $\forall x_j \in X_j$, $\theta_i^\alpha(x_j) \in X_i^\alpha$ (correspondingly $\theta_j^\beta: X_i \to X_j^\beta$ that $\forall x_i \in X_i$, $\theta_j^\beta(x_i) \in X_j^\beta$). We denote by

$$\Theta_i = \{\theta_i^\alpha: X_j \to X_i^\alpha, \alpha = 1, \dots, \chi_i\}$$

and correspondingly

$$\Theta_j = \{\theta_j^\beta: X_i \to X_j^\beta, \beta = 1, \dots, \chi_j\}$$

the sets of the informational extended strategies of the player i and correspondingly j, in the $i \leftrightarrow j$

[1] Here the j≠i players do not know exactly the form of the function H_p.

[2] Players i and j not knowing the informationally extended strategies chosen by their partners will lead to the impossibility of constructing their utility functions.

informational extended game. The $\left(\theta_i^\alpha \in \Theta_i, \theta_j^\beta \in \Theta_j, x_{[-ij]} \in X_{[-ij]}\right)$ denotes the strategy profile in the double-way informational extended game.

Our goal is to determine the values of the payoff functions for the players so that the concrete value of the payoff will be determined using the payoffs of the players in the non informational extended game (parent game) Γ. This problem is more difficult than in the one-way informational extended game. Let player i and j choose the informational extended strategy $\theta_i^\alpha \in \Theta_i$ and $\theta_j^\beta \in \Theta_j$ respectively. Denote by $X\left(\theta_i^\alpha, \theta_j^\beta\right) \subseteq X$ the set of the strategy profiles of the players in the game Γ "generated" by the informational extended strategy θ_i^α and θ_j^β. It is easy to show that

$$X\left(\theta_i^\alpha, \theta_j^\beta\right) = \left(gr\theta_i^\alpha \bigcup gr\theta_j^\beta\right) \bigcup X_{[-ij]}.$$

Here $gr\theta_i^\alpha$ corresponding $gr\theta_j^\beta$. is graph of the function θ_i^α and θ_j^β. We mention that there exist the strategies θ_i^α and θ_j^β such that $X\left(\theta_i^\alpha, \theta_j^\beta\right) = \emptyset$, i.e. not for any couple $\left(\theta_i^\alpha, \theta_j^\beta\right)$ we can construct the carry out strategy profile in the non informational extended game. Let us assume that the players want maximize their payoffs. Then we can define payoff functions of the player as following

$$\mathcal{H}_p\left(\theta_i^\alpha, \theta_j^\beta, x_{[-ij]}\right) = \begin{cases} \max\limits_{(x_i, x_j) \in \left[gr\theta_i^\alpha \cap gr\theta_j^\beta\right]} H_p\left(x_i, x_j, x_{[-ij]}\right) & \text{if } X\left(\theta_i^\alpha, \theta_j^\beta\right) \neq \emptyset, \\ -\infty & \text{if } X\left(\theta_i^\alpha, \theta_j^\beta\right) = \emptyset. \end{cases}$$

Finally the normal form of the double-way i ↔ j informational extended game will be

$$\Gamma(\mathrm{i} \leftrightarrow \mathrm{j}) = \left\langle \mathrm{I}, \Theta_\mathrm{i}, \Theta_\mathrm{j}, \{\mathrm{X}\}_{\mathrm{p} \in \mathrm{I} \setminus \{\mathrm{i},\mathrm{j}\}}, \{\mathcal{H}_\mathrm{p}\}_{\mathrm{p} \in \mathrm{I}} \right\rangle.$$

3. Properties of the Nash Equilibrium Profiles in the Informational Extended Games

In this paragraph we study the properties of the strategy profiles of the non informational extended game Γ which can be generated by the Nash equilibrium profiles of the informational extended games $\Gamma(j \to i)$ and $\Gamma(i \leftrightarrow j)$.

Theorem 3.1. *The following relations are true:*

$$NE[\Gamma] \subseteq \bigcup_{\substack{\left(\theta_i^k, x_{[-j]}\right) \in NE[\Gamma(\mathrm{j} \to \mathrm{i})] \\ k = 1, \ldots, \chi_i}} X\left(\theta_i^k\right) \subseteq \bigcup_{\substack{\left(\theta_i^\alpha, \theta_j^\beta, x_{[-ij]}\right) \in NE[\Gamma(i \leftrightarrow j)] \\ \alpha = 1, \ldots, \chi_i, \beta = 1, \ldots, \chi_j}} X\left(\theta_i^\alpha, \theta_j^\beta\right).$$

Proof. It can easily be proved that there are $|\mathrm{X}_\mathrm{j}|$ informational extended strategies $\hat{\theta}_\mathrm{i}^\alpha \in \Theta_\mathrm{i}$, $\alpha =$

$\overline{1,|X_j|}$ of the player i, respectively $|X_i|$ informational extended strategies $\hat{\theta}_j^\beta \in \Theta_j$, $\beta = \overline{1,|X_i|}$ of the player j, such that $X\left(\hat{\theta}_i^\alpha, \theta_j^\beta\right) = X\left(\hat{\theta}_i^\alpha\right)$ for all $\theta_j^\beta \in \Theta_j$, respectively $X\left(\theta_i^\alpha, \hat{\theta}_j^\beta\right) = X\left(\hat{\theta}_j^\beta\right)$ for all $\theta_i^\alpha \in \Theta_i$. Let $\tilde{x} \in NE[\Gamma]$ then $H_p(\tilde{x}) \geq H_p(x_p, \tilde{x}_{[-p]})$ for all $x_p \in X_p$ and for all $p \in I$.. We consider the following informational extended strategy $\hat{\theta}_i^\alpha(x_j) = \tilde{x}_i$ for all $x_j \in X_j$, and then $H_i\left(\tilde{x}_i, \tilde{x}_{[-i]}\right) \geq H_i(x_i, \tilde{x}_{[-i]})$ $\forall x_i \in X_i$. So in this case we have $\tilde{x} \in X\left(\hat{\theta}_i^\alpha\right)$ since $X\left(\hat{\theta}_i^\alpha\right) = gr\hat{\theta}_i^\alpha \cup X_{[-ij]}$. Now we have to prove that $\left(\hat{\theta}_i^\alpha, \tilde{x}_{[-i]}\right) \in NE[\Gamma(\mathrm{j} \to \mathrm{i})]$, that is $H_i\left(\hat{\theta}_i^\alpha(\tilde{x}_j), \tilde{x}_j, \tilde{x}_{-[i,j]}\right) = H_i\left(\tilde{x}_i, \tilde{x}_j, \tilde{x}_{-[i,j]}\right) \geq H_i\left(\theta_i^\alpha(\tilde{x}_j), \tilde{x}_j, \tilde{x}_{-[i,j]}\right)$ for all $\alpha = \overline{1, \chi_i}$, $H_j\left(\tilde{x}_i, \tilde{x}_j, \tilde{x}_{-[i,j]}\right) \geq H_j\left(\tilde{x}_i, x_j, \tilde{x}_{-[i,j]}\right)$ $\forall x_j \in X_j$, $H_p\left(\tilde{x}_i, \tilde{x}_j, \tilde{x}_{-[i,j]}\right) \geq H_p\left(\tilde{x}_i, \tilde{x}_j, x_p, \tilde{x}_{-[i,jp]}\right)$ $\forall x_p \in X_p$ for all $p \in I\{i, j\}$. But these inequalities are true since $\tilde{x} \in NE[\Gamma]$. So we proof that there is the strategy profile $\left(\hat{\theta}_i^\alpha, \tilde{x}_{[-i]}\right) \in NE[\Gamma(\mathrm{j} \to \mathrm{i})]$, respectively $\left(\hat{\theta}_j^\beta, \tilde{x}_{[-j]}\right) \in NE[\Gamma(\mathrm{i} \to \mathrm{j})]$ such that $\tilde{x} \in X\left(\hat{\theta}_i^\alpha\right)$, respectively $\tilde{x} \in X\left(\hat{\theta}_j^\beta\right)$. Consider the informational extended strategy of player i $\hat{\theta}_i^\alpha(x_j) = \tilde{x}_i$ $\forall x_j \in X_j$, and for player j the informational extended strategy of player $\hat{\theta}_j^\beta(x_i) = \tilde{x}_j$ $\forall x_i \in X_i$, and then $H_i\left(\tilde{x}_i, \tilde{x}_{[-i]}\right) \geq H_i(x_i, \tilde{x}_{[-p]})$ $\forall x_i \in X_i$ and $H_j\left(\tilde{x}_j, \tilde{x}_{[-j]}\right) \geq H_j(x_j, \tilde{x}_{[-j]})$ $\forall x_i \in X_i$. Based on the above we obtained that $\tilde{x} \in X\left(\hat{\theta}_i^\alpha\right) \cap X\left(\hat{\theta}_j^\beta\right)$. Since $gr\hat{\theta}_i^\alpha \cap gr\hat{\theta}_j^\beta = \left(\tilde{x}_i, \tilde{x}_j\right)$ and $\tilde{x} \in NE[\Gamma]$ we can prove that $\max\limits_{(x_i,x_j)\in\left[gr\tilde{\theta}_i^\alpha \cap gr\tilde{\theta}_j^\beta\right]} H_i\left(x_i, x_j, \tilde{x}_{[-ij]}\right) \geq$

$\max\limits_{(x_i,\tilde{x}_j)\in\left[gr\theta_i^\alpha \cap gr\tilde{\theta}_j^\beta\right]} H_i\left(x_i, \tilde{x}_j, \tilde{x}_{[-ij]}\right)$, $\max\limits_{(\tilde{x}_i,\tilde{x}_j)\in\left[gr\tilde{\theta}_i^\alpha \cap gr\tilde{\theta}_j^\beta\right]} H_j\left(\tilde{x}_i, \tilde{x}_j, \tilde{x}_{[-ij]}\right) \geq$

$\max\limits_{(\tilde{x}_i,x_j)\in\left[gr\tilde{\theta}_i^\alpha \cap gr\theta_j^\beta\right]} H_j\left(\tilde{x}_i, x_j, \tilde{x}_{[-ij]}\right)$, $\max\limits_{(\tilde{x}_i,\tilde{x}_j)\in\left[gr\tilde{\theta}_i^\alpha \cap gr\tilde{\theta}_j^\beta\right]} H_p\left(\tilde{x}_i, \tilde{x}_j, \tilde{x}_{[-ij]}\right) \geq$

$\max\limits_{(\tilde{x}_i,\tilde{x}_j)\in\left[gr\tilde{\theta}_i^\alpha \cap gr\tilde{\theta}_j^\beta\right]} H_p\left(\tilde{x}_i, \tilde{x}_j, x_p, \tilde{x}_{-[i,jp]}\right)$ $\forall p \in I\backslash\{i, j\}$.

So we prove that $\left(\hat{\theta}_i^\alpha, \hat{\theta}_j^\beta, \tilde{x}_{[-ij]}\right) \in NE[\Gamma(i \leftrightarrow j)]$. Now we have that

$$NE[\Gamma] \subseteq \bigcup_{\substack{(\theta_i^k, x_{[-j]}) \in NE[\Gamma(\mathrm{j}\to\mathrm{i})] \\ k=1,\dots,\chi_i}} X\left(\theta_i^k\right) \text{ and } NE[\Gamma] \subseteq \bigcup_{\substack{\left(\theta_i^\alpha, \theta_j^\beta, x_{[-ij]}\right) \in NE[\Gamma(i\leftrightarrow j)] \\ \alpha=1,\dots,\chi_i, \beta=1,\dots,\chi_j}} X\left(\theta_i^\alpha, \theta_j^\beta\right).$$

Finaly it is easy to see that under the above

$$\bigcup_{\substack{(\theta_i^k, x_{[-j]}) \in NE[\Gamma(\mathrm{j}\to\mathrm{i})] \\ k=1,\dots,\chi_i}} X\left(\theta_i^k\right) \subseteq \bigcup_{\substack{\left(\theta_i^\alpha, \theta_j^\beta, x_{[-ij]}\right) \in NE[\Gamma(i\leftrightarrow j)] \\ \alpha=1,\dots,\chi_i, \beta=1,\dots,\chi_j}} X\left(\theta_i^\alpha, \theta_j^\beta\right)$$

which proves the theorem. ■

This theorem shows that the set of the equilibrium profiles in the non informational extended games Γ *is containing in the set of non informational extended strategy profiles generating by the Nash equilibrium profiles in the informational extended game*. Moreover, the set of non informational extended strategy profiles generating by the Nash equilibrium profiles in the game $\Gamma(j \to i)$ or $\Gamma(i \to j)$ is contained in the set of non informational extended strategy profiles generating by the Nash equilibrium profiles in the game $\Gamma(i \leftrightarrow j)$.

4. Informational non-Extended Game Generated by the Informational Extended Strategies

In this paragrath we study the case when the informational strategies of the players have already been chosen and so appears the necessity to study the informational non-extended game generated by the chosen informational extended strategies (Hancu 2012). These games differ in: a) the sets of the strategies that are the subsets of the sets of strategies in the initial non-extended informational game; b) how the payoff functions of the players will be constructed.

Let the payoff functions of the players be defined as $\widetilde{H}_p: \prod_{p \in I} X_p \to R$ where for all $x_i \in X_i, x_j \in X_j, x_{[-ij]} \in X_{[-ij]}$,

$$\widetilde{H}_p\left(x_i, x_j, x_{[-ij]}\right) \equiv H_p\left(\theta_i\left(x_j\right), \theta_j(x_i), x_{[-ij]}\right).$$

The game with the following normal form $\Gamma\left(\theta_i, \theta_j\right) = \left\langle I, \{X_p\}_{p \in I}, \{\widetilde{H}_p\}_{p \in I}\right\rangle$ will be called *informational non-extended game generated by the informational extended strategies* θ_i *and* θ_j. The game $\Gamma\left(\theta_i, \theta_j\right)$ is played as follows: independently and simultaneously each player $p \in I$ chooses the informational non-extended strategy $x_p \in X_p$, after that the players i and j calculate the value of the informational extended strategies $\theta_i(x_j)$ and $\theta_j\left(x_j\right)$, after that each player calculates the payoff values $H_p\left(\theta_i(x_j), \theta_j(x_i), x_{[-ij]}\right)$, and with this the game is finished.

To all strategy profiles $\left(x_i, x_j, x_{[-ij]}\right)$ in the game Γ the following realization $\left(\theta_i(x_j), \theta_j(x_i), x_{[-ij]}\right)$ in terms of the informational extended strategies will correspond. We denote by $NE\left[\Gamma\left(\theta_i, \theta_j\right)\right]$ the set of Nash equilibrium profiles of the game $\Gamma\left(\theta_i, \theta_j\right)$.

Theorem 4.1. *Let the game* $\Gamma\left(\theta_i, \theta_j\right)$ *satisfy the following conditions:*

1. *the* X_p *is a non-empty compact and convex subset of the finite-dimensional Euclidean space for all* $p \in I$;
2. *the functions* θ_i *(correspondingly* θ_j*) are continuous on* X_j *(correspondingly on* X_i*) and the*

functions H_p are continuous on X for all $p \in I$;

3. *the functions θ_i (correspondingly θ_j) are quasi-concave on X_j (correspondingly on X_i), the functions H_p are quasi-concave on $X_p, p \in I \backslash \{i, j\}$ and monotonically increasing on $X_i \times X_j$.*

Then $NE[\Gamma(\theta_i, \theta_j)] \neq \emptyset$.

Proof. We define the following correspondence (point-to-set mapping) $Br: X \to X$ such that

$$Br(x) = \left(Br_1(x_{[-1]}), \ldots, Br_i(x_{[-i]}), \ldots, Br_n(x_{[-n]})\right)$$

then if $x^* \in Br(x^*)$, then $x_i^* \in Br_i(x_{[-i]}^*)$ for all $i \in I$ and hence $x^* \in NE$. According to the Tikhonov's theorem: $X = \prod_{p \in I} X_p$ is compact, and according to conditions 1) and 2) for all $x_{[-i]}$ the set $Br_i(x_{[-i]})$ is non-empty. According to condition 3) $Br_i(x_{[-i]})$ is also convex because $\widetilde{H}_i$ is quasi-concave on X_i. Hence the set $Br(x)$is nonempty convex and compact for all $x \in X$. The mapping Br has a closed graph because each function $\widetilde{H}_p$ is continuous on X for all $p \in I$. Hence by Kakutani's theorem, the set-valued mapping Br has a fixed point. As we have noted, any fixed point is a Nash equilibrium. ■

To exemplify what was mentioned above, in the following paragraph we will analyze the duopoly type games.

4.1 Informational extended strategies in the duopol games

In this paragraph we study the *Cournot's Duopol* games generated by the informational extended strategies. There are two firms operating in a limited market. Market production is $p(Q) = \begin{cases} a - Q & \text{for} \quad Q < a \\ 0 & \text{for} \quad Q \geq a \end{cases}$, where $Q = q_1 + q_2$ for two firms, *a*> *0* represents the volume of products of type q_1 and q_2 that is in the market, *p* represents the price of the good. Both firms will receive profits derived from a simultaneous decision made by both on how much to produce, and also based on their cost functions: $C_i(q_i) = cq_i + C_i$ where $c \leq a$ represents the marginal cost (consumption for both firms) and $C_i \leq a$ the fixed cost. Both firms simultaneously choose the quantity of the goods that they offer on the market.

We will model the activity described in the example using the game theory. Since both firms simultaneously choose the quantity of good that they offer on the market and do not cooperate, then we will use the non-cooperative static game. The normal form of these game is $\Gamma = \langle I = \{1,2\}, X = [0, N_1], Y = [0. N_2], H_1, H_2 \rangle$, where I is the set of players: firm 1 and firm 2; X, Y are the sets of strategies (the *x* and *y* represents the volume of products of type q_1 and q_2 that is in the market) of the players, $H_1(x, y) = x(a - x - y - c) - C_1$, $H_2(x, y) = y(a - x - y - c) - C_2$ reprezent the payoffs of the player 1 and 2.

The Nash equilibrium profile is $(x^*, y^*) = \left(\frac{a-c}{3}, \frac{a-c}{3}\right)$ and firms will receive the following profits $H_i(x, y) = \frac{(a-c)^2}{2} - C_i$ if $x + y < a$ and $a > 2c$. From the property of the equilibrium profile, we obtain that none of the firms is unilaterally convenient to offer on the market a volume of goods other than the value $\frac{a-c}{3}$, otherwise the firm will lose income.

We analyze different forms of informational extended strategies.

A) In this case, the sets of strategies of the player *1*, respectively the player *2*, will be $\Theta_1 = \{\theta_1: Y \to X$ such that

$$\forall y \in Y, \theta_1(y) = \underset{x\in X}{\operatorname{argmax}} H_1(x, y)\},$$

respectiv $\Theta_2 = \{\theta_2: X \to Y$ such that

$$\forall x \in X, \theta_2(x) = \underset{y\in Y}{\operatorname{argmax}} H_2(x, y)\}.$$

Players do not know which function θ_1 or θ_2 will be chosen by the partners (but this is not significant because the sets Θ_i contain one element). We determine the functions $\theta_1(y)$ si $\theta_2(x)$. The functions $H_1(x, y)$ and $H_2(x, y)$ are concave, then using the necessary conditions of the functions extrem we obtane that $\underset{x\in X}{\operatorname{argmax}} H_1(x, y)\} = \left\{x \in X: x = \frac{a-y-c}{2}\right\}$ and $\underset{y\in Y}{\operatorname{argmax}} H_2(x, y)\} = \left\{y \in Y: y = \frac{a-x-c}{2}\right\}$.

Then $\theta_1(y) = \frac{a-y-c}{2}$ such that $\forall y \in Y$, $\frac{a-y-c}{2} \in X$ (but no $\theta_1(y) = a - 2x - c$) and $\theta_2(x) = \frac{a-x-c}{2}$ such that $\forall x \in X, \frac{a-x-c}{2} \in Y$ (but no $\theta_2(x) = a - 2y - c$). So, $\widetilde{H}_1\big(\theta_1(y), \theta_2(x)\big) = \frac{1}{2}[y(a - y - x - c) + x(a - c)] - C_1$. Similarly we will get $\widetilde{H}_2\big(\theta_1(y), \theta_2(x)\big) = \frac{1}{2}[x(a - y - x - c) + y(a - c)] - C_2$. Then Nash equilibrium profile is $(x^*, y^*) = (a - c, a - c)$, which does not coincide with the situation in the classical case, and each company's payoffs will be

$$\widetilde{H}_i(\theta_1(y^*), \theta_2(x^*)) = \frac{1}{2}(a - c)[(c - a) + (a - c)^2] - C_i.$$

So if every firms knows what the volume of goods is going to launch on the market, and each of them takes advantage of this, choose its strategies based on the functions of type $\theta_1(y) = \underset{x\in X}{\operatorname{argmax}} H_1(x, y)$, respectively $\theta_2(x) = \underset{y\in Y}{\operatorname{argmax}} H_2(x, y)$, then the volume of goods released on the market, based on Nash equilibrium profile, is higher than when firms do not possess that information, and the payoffs is much higher.

B) Analise the case when "firm *1* knows what volume of goods firm *2* will be launched on market, while firm *2* does not know what volume of goods will launch firm *1*". Consider the case when the firm 1 informational extended strategy is a function $\theta_1(y) = \underset{x\in X}{\operatorname{argmax}} H_1(x, y)$ $\forall y \in Y$, but on the firm *2* is the function $\theta_2(x) = y \;\; \forall x \in X$. The payoffs will be $\widetilde{H}_1(\theta_1(y), \theta_2(x)) = \left(\frac{a-y-c}{2}\right)^2 - C_1$ and $\widetilde{H}_2(\theta_1(y), \theta_2(x)) = y\left(\frac{a-y-c}{2}\right) - C_2$. In this case, the player's *1* payoff does not depend on the x strategy. We determine equilibrium profiles for this game. This is where the notion of equilibrium is already lost. Then the solution will be determined accordingly. Player 2 determines y^* that realizes $\max_{y \in Y}\left\{y\left(\frac{a-y-c}{2}\right)\right\}$, and so $y^* = \frac{a-c}{2}$. Then $x^* = \theta_1(y^*) = \theta_1(\frac{a-c}{2}) = \frac{a-c}{4}$. We note that $\left(\frac{a-c}{4}, \frac{a-c}{2}\right)$ profile coincides with the Stachelberg equilibrium profile in which player 2 makes the first move.

5. Converting the Two Persons Game with Informational Extended Strategies to Bayesian Game

Consider the two person $\Gamma = \langle \mathrm{X}, \mathrm{Y}, ; \mathrm{H_i}: \mathrm{X} \times \mathrm{Y} \to \mathbb{R}, \mathrm{i} = 1,2 \rangle$ informational extended game with the following sets of the informational extended strategies

$$\Theta_1 = \{\theta_1^j: Y \to X | \forall y \in Y, \theta_1^j(y) \in X, j = \overline{1, m_1}\}$$

of the player 1 and

$$\Theta_2 = \{\theta_2^k: X \to Y | \forall x \in X, \theta_2^k(x) \in Y, k = \overline{1, m_2}\}.$$

The payoff functions of the players are defined as following: for all $\theta_1 \in \Theta_1$, $\theta_2 \in \Theta_2$, $\mathcal{H}_i(\theta_1, \theta_2) = H_i(\theta_1(y), \theta_2(x))$ for all $x \in X, y \in Y$. The game is played as follows: independently and simultaneously each player $i \in I = \{1,2\}$ chooses the informational extended strategy $\theta_1 \in \Theta_1$ and $\theta_2 \in \Theta_2$ (and players do not know what kind of the informational extended strategy will be chosed by each other's), after that the players 1 and 2 calculate the value of the payoff values $H_i(\theta_1(y), \theta_2(x))$, and with this the game is finished.

Remark 5.1. *The game described above will be denoted by*[3] $Game(1 \leftrightarrow 2)$ *and is the game with* ***incomplete information*** *because players do not know what kind of the informational extended strategy* $\theta_1 \in \Theta_1$ *will be chosen by the player* 1 *(for example) and so the player* 1 *generates the uncertainty of the player* 2 *about the complete structure of the payoff function* $H_2(\theta_1(y), \theta_2(x))$ *in the game with non informational extended strategies. So the players do not know exactly the structure of yours payoff functions and the game is in the incomplete information.*

Denote also by

[3] We note that this notation does not denote the normal form of the game, since we do not know exactly the utility functions of the players

$$\tilde{X}_j = \{\tilde{x}_j \in X: \tilde{x}_j = \theta_1^j(y), \forall y \in Y\}$$

and

$$\tilde{Y}_k = \{\tilde{y}_k \in Y: \tilde{y}_k = \theta_2^k(x), \forall x \in X\}$$

the set of all range of the informational extended strategy θ_1^j of the player 1 and θ_2^k of the player 2. The sets $\tilde{X}_j$ and $\tilde{Y}_k$ are the sets of informational non extended strategies generated by the informational extended strategies of the player 1 and 2 respectively. According to Harsanyi –Selten (Harsanyi 1998) principle we can reduce the analysis of a game with incomplete information to the analysis of a game with complete (but imperfect) information, which is fully accessible to the usual analytical tools of game theory.

So to solve the game $Game(1 \leftrightarrow 2)$ we must do the following step-intervals:

1. Construct the Bayesian game

$$\Gamma_B = \langle I = \{1,2\}, S_1(\Delta_1), S_2(\Delta_2), \Delta_1, \Delta_2, p, q, \widetilde{H}_1, \widetilde{H}_1 \rangle$$

that corresponds (is associated) to the game $Game(1 \leftrightarrow 2)$. The normal form must consisting of the following.

- A set of possible actions for the player 1 is X, a for player 2 is Y.
- Because players do not know what kind of the informational extended strategy will be chosen by the other player, then the uncertainty of the player 1 about the own payoff function structure is generated by the player 2 selected informational extended strategy and respectively, the uncertainty of the player 2 about the own payoff function structure is generated by the player 1 selected informational extended strategy. The set of types for player 1 (player 2) is $\Delta_1 = \{\delta_1^j, j \in J_1\}(\Delta_2 = \{\delta_2^k, k \in J_2\})$. In other words, the player 1(player 2) is of the type δ_1^j (of the type δ_2^k) if he generates to the player 2 (to the player 1) payoff structure uncertainty, selecting the $\theta_1^j \in \Theta_1$ ($\theta_2^k \in \Theta_2$) informational extended strategy.
- The probability function $p: \Delta_1 \rightarrow \Omega(\Delta_2)$ of the player 1, respectively $q: \Delta_2 \rightarrow \Omega(\Delta_1)$ of the player 2, means the following: if the player 1(player 2) chooses the informational extended strategy θ_1^j (strategy θ_2^k), then he believes that the player 2 (player 1) with the $p(\delta_2^k/\delta_1^j) = \frac{p(\delta_2^k \cap \delta_1^j)}{p(\delta_1^j)}$ (respectively $q(\delta_1^j/\delta_2^k) = \frac{q(\delta_1^j \cap \delta_2^k)}{q(\delta_2)}$) chooses the informational extended strategy θ_2^k (strategy θ_1^j).
- The set of the strategies of the players are the set of all range of the informational extended strategies of the players

$$S_1(\Delta_1) = \{\tilde{x}_j \in X: \tilde{x}_j = \theta_1^j(y), \forall y \in Y, \forall j \in J_1\} \equiv \{\tilde{X}_j, j = 1, \ldots, m_1\},$$

and

$$S_2(\Delta_2) = \{\tilde{y}_k \in Y: \tilde{y}_k = \theta_2^k(x), \forall x \in X, , \forall k \in J_2\} \equiv \{\tilde{Y}_k, k = 1, \ldots, m_2\}.$$

So if the player 1, for example, is of the type j, i.e. he chooses the informational extended strategy θ_1^j, then the set of the informational non extended strategies, generated by θ_1^j is $\tilde{X}_j$.

- The payoff functions of the player is defined as following

$$\tilde{H}_1: S_1(\Delta_1) \times S_2(\Delta_2) \times \Delta_1 \times \Delta_2 \to R, \tilde{H}_2: S_1(\Delta_1) \times S_2(\Delta_2) \times \Delta_1 \times \Delta_2 \to R.$$

More exact, for all fixed $s_1 \in S_1(\Delta_1)$ and $s_2 \in S_2(\Delta_2)$,

$$\tilde{H}_1(s_1(\cdot), s_2(\cdot), \delta_1^j, \delta_2^k) = H_1(\tilde{x}_j, \tilde{y}_k, \delta_1^j, \delta_2^k),$$

and

$$\tilde{H}_2(s_1(\cdot), s_2(\cdot), \delta_1^j, \delta_2^k) = H_2(\tilde{x}_j, \tilde{y}_k, \delta_1^j, \delta_2^k)$$

for all $\tilde{x}_j \in \tilde{X}_j, \tilde{y}_k \in \tilde{Y}_k, j = \overline{1, m_1}, k = \overline{1, m_2}$.

2. For game Γ_B construct the Selten-Harsanyi game

$$\Gamma_B^* = \left\langle J, \{R_j\}_{j\in J}, \{U_j\}_{j\in Jj} \right\rangle$$

with complete and imperfect information on the sets of the non-informational extended strategies. Denote by $J = \{j = (i, \delta_i^j), i = 1,2, j = 1, \ldots, m_1 + m_2\}$ the set of type-players that is equal to the sets of all informational extended strategies of the players $J = J_1 \cup J_2$. The strategy of the type-player j is $r_j = \begin{cases} \tilde{x}_j \in \tilde{X}_j & j \in J_1, \\ \tilde{y}_j \in \tilde{Y}_j & j \in J_2 \end{cases}$ and means the following: if player is of type $j = (i, \delta_i^j)$ (i.e. player $i, i = 1,2$, chooses the informational extended strategy θ_i^j), then the strategy will be equal to value of the informational extended strategy $x^j = \theta_i^j(y)$ for a fixed value of the non extended strategy $y \in Y$. The sets of pure strategies of the players will be $R_j = \begin{cases} \tilde{X}_j & j \in J_1, \\ \tilde{Y}_j & j \in J_2. \end{cases}$, and $R = \prod_{j=1}^{m_1+m_2} R_j$. For all type-player $j = (1, \delta_1^j),\ j \in J_1$, payoff function $U_j: \tilde{X}_j \times \left({}_{k\in J_2} \tilde{Y}_k \right)$ is defined as following

$$U_j(r_j, \{r_k\}_{k\in J_2}) = U_j(\tilde{x}_j, \{\tilde{y}_k\}_{k\in J_2}) = \sum_{k\in J_2} p(\delta_2^k | \delta_1^j) H_1(\tilde{x}_j, \tilde{y}_k), \forall \tilde{x}_j \in \tilde{X}_{j,} \tilde{y}_k \in \tilde{Y}_k.$$

In similar mode for all type-player $j = (2, \delta_2^j),\ j \in J_2$ payoff function $U_j: \left({}_{k\in J_1} \tilde{X}_k \right) \times \tilde{Y}_j$ is defined as following

$$U_j\left(\{r_k\}_{k\in J_1}, r_j\right) = U_j\left(\{\tilde{x}_k\}_{k\in J_1}, \tilde{y}_j\right) = \sum_{k\in J_1} q\left(\delta_1^k|\delta_2^j\right) H_2\left(\tilde{x}_k, \tilde{y}_j\right), \forall \tilde{x}_k \in \tilde{X}_k, \tilde{y}_j \in \tilde{Y}_j.$$

These utility functions have the following meaning. If, for example, the player 1, had chosen information extended strategy θ_1^j, which also means he has a type-player $j = (1, \delta_1^j)$, and with the probability $p(\delta_2^k|\delta_1^j)$ he assumes that the player 2 will choose the information extended strategy θ_2^k, i.e. as we have the type player $k = (2, \delta_2^k)$, for all $k \in J_2$, then for all information not extended strategy $x \in X$ and $y \in Y$, average value of the payoff will be

$$\sum_{k\in J_2} p\left(\delta_2^k|\delta_1^j\right) H_1\left(\tilde{x}_j, \tilde{y}_k\right) \equiv \sum_{k\in J_2} p\left(\delta_2^k|\delta_1^j\right) H_1\left(\theta_1^j(y), \theta_2^k(x)\right)$$

3. Determine Nash equilibrium profiles in the game Γ_B^* that is the Bayes-Nash equilibrium in the game Γ_B.
4. As the solution of the game $\text{Game}(1 \leftrightarrow 2)$ we will consider the non informational extended strategy profile (x^*, y^*) which is generated by the Nash strategy profile in the game Γ_B^*.

Denote by $BE[\Gamma_B]$ the set of all Bayes-Nash strategies profile of the game Γ_B. Really, solving the game Γ_B is difficult, because it is transformed in the two level dynamic game. On the first level player Nature chooses the informational extended strategies of the players, for example $\left(\theta_1^j, \theta_2^k\right)$, and on the second level each player choose the informational non extended strategies from the set $\tilde{X}_j$ (player 1) and from the set $\tilde{Y}_k$ (player 2). The games $\Gamma\left(\theta_1^j, \theta_2^k\right)$ are the subgames in the dynamic game defined above. Here we do not investigate such method to solve informational extended game.

The game Γ_B^* is played as follows: for all fixed probabilities $p(\delta_2^k|\delta_1^j)$ and $q\left(\delta_1^k|\delta_2^j\right)$, independently and simultaneously each type-player $j = (i, \delta_i^j)$ chooses the strategy $r_j \in R_j$, after that each player calculates the payoff using the functions $U_j\left(r_j, \{r_k\}_{k\in J_2}\right)$ or $U_j\left(\{r_k\}_{k\in J_1}, r_j\right)$ and whereupon the game is finished. In other words, because strategies r_j are defined by the informational non extended strategies from sets X and Y, for all $y \in Y$ (respectively for all $x \in X$) type-player $j = (1, \delta_1^j)$ (respectively type-player $k = (2, \delta_2^k)$) chooses the strategy $\tilde{x}_j = \theta_1^j(y)$ (respectively $\tilde{y}_k = \theta_2^k(x)$), calculates the payoff values using the functions $U_j\left(\tilde{x}_j, \{\tilde{y}_k\}_{k\in J_2}\right)$ (respectively $(U_j\left(\{\tilde{x}_k\}_{k\in J_1}, \tilde{y}_j\right))$) and with this the game is finished.

We introduce the following definition.

Definition 5.1. *Strategy profile* $r^* = \left(r_1^*, \dots, r_j^*, \dots r_{|J|}^*\right)$ *is the Nash equilibrium in the game* Γ_B^* *if an only if the following conditions are fulfilled*:

$$\begin{cases} U_j\left(r_j^*, \{r_k^*\}_{k\in J_2}\right) \geq U_j\left(r_j, \{r_k^*\}_{k\in J_2}\right) \text{ for all } j \in J_1, \\ U_j\left(\{r_k^*\}_{k\in J_1}, r_j^*\right) \geq U_j\left(\{r_k^*\}_{k\in J_1}, r_j\right) \text{ for all } j \in J_2. \end{cases}$$

We get for all fixed probabilities $p(\cdot), q(\cdot)$ that strategy profile $\left(\{\tilde{x}_j^*\}_{l\in J_1}, \{\tilde{y}_k^*\}_{k\in J_2}\right)$ is Nash equilibrium for the game $\Gamma_B^* = \left\langle J, \{R_j\}_{j\in J}, \{U_j\}_{j\in J}\right\rangle$ if and only if the $|J_1| + |J_2|$ conditions are fulfilled:

$$\begin{cases} \sum_{k\in J_2} p(\delta_2^k|\delta_1^j) H_1\left(\tilde{x}_j^*, \tilde{y}_k^*\right) \geq \sum_{k\in J_2} p\left(\delta_2^k\middle|\delta_1^j\right) H_1\left(\tilde{x}_j, \tilde{y}_k^*\right) \text{ for all } \tilde{x}_j \in \tilde{X}_\mathrm{j}, j \in J_1, \\ \sum_{k\in J_1} p(\delta_1^j|\delta_2^k) H_1\left(\tilde{x}_j^*, \tilde{y}_k^*\right) \geq \sum_{k\in J_2} p(\delta_1^j|\delta_2^k) H_2\left(\tilde{x}_j^*, \tilde{y}_k\right) \text{ for all } \tilde{y}_k \in \tilde{Y}_\mathrm{j}, k \in J_2. \end{cases}$$

Denote by $NE[\Gamma_B^*]$ the set of all Nash equilibrium strategies profile in the game Γ_B^*. The relation between the Nash equilibrium in the Harsanyi game Γ_B^* and the equilibrium at the Bayesian game Γ_B was given by Harsanyi theorem: *The set of Nash equilibria of the game Γ_B^* is identical to the set of Bayesian equilibria of the game Γ_B*.

Let strategy profile$\left(\{\tilde{x}_j^*\}_{j\in J_1}, \{\tilde{y}_k^*\}_{k\in J_2}\right) \in NE[\Gamma_B^*]$ then we introduce the following definition.

Definition 5.2. *For all fixed probabilities $p(\cdot), q(\cdot)$ strategy profile $(x^*, y^*) \equiv (x^*(p), y^*(q))$ $x^* \in X, y^* \in Y$, for which the following conditions*

$$\begin{cases} \tilde{x}_j^* = \theta_1^j(y^*) \ \ \forall j \in J_1 \\ \tilde{y}_k^* = \theta_2^k(x^*) \ \ \forall k \in J_2 \end{cases}$$

are fulfilled, is called the Bayes-Nash equilibrium profile in non informational extended strategies of the game Γ generated by the $\left(\{\tilde{x}_j^*\}_{j\in J_1}, \{\tilde{y}_k^*\}_{k\in J_2}\right) \in NE[\Gamma_B^*]$.

Denote by $BN[G(1 \leftrightarrow 2)]$ the set of all Bayes-Nash equilibrium profiles in the game $Game(1 \leftrightarrow 2)$. So, such as a solutions of the informational extended games $Game(1 \leftrightarrow 2)$ we consider the informational non extended Bayes-Nash equilibrium profiles $(x^*, y^*) \equiv (x^*(p), y^*(q))$ for which the relations from Definition 5.2 are fulfilled for all fixed probabilities $p(\cdot)$ and $q(\cdot)$ of the believes about the choice of the informational extended strategy of the other player.

For all $j = 1,2, k = 1,2$, denote by ϑ_1^j (respectively ϑ_2^k) an inverse function of the θ_1^j (respectively θ_2^k). If informational extended strategies $\theta_1^j(y)$ and $\theta_2^k(x)$ for all $j = 1,2, k = 1,2$ are bijective, then there are the inverse functions ϑ_1^j and ϑ_2^k for all $j = 1,2, k = 1,2$ and, so, we have that for all $\left(\{\tilde{x}_j^*\}_{l\in J_1}, \{\tilde{y}_k^*\}_{k\in J_2}\right) \in NE[\Gamma_B^*]$ there is (x^*, y^*) such that $y^* = \vartheta_1^j(\tilde{x}_j^*)\ \forall j \in J_1$ and $x^* = \vartheta\left(\tilde{y}_j^*\right) \forall k \in J_2$. We can proof the following theorem.

Theorem 5.1. *Let the game Γ satisfy the following conditions:*

1. *X and Y are a non-empty compact and convex subsets of the finite-dimensional Euclidean space;*
2. *the functions $\theta_1^j, \forall j \in J_1$, and θ_2^k, $\forall k \in J_2$, are continuous on Y (respectively on X) and the functions H_1, H_2 are continuous on $X \times Y$;*
3. *the functions $\theta_1^j, \forall j \in J_1$, (respectively θ_2^k, $\forall k \in J_2$), are quasi-concave on Y (respectively on X), the functions H_1 (respectively H_2) are quasi-concave on X (respectively Y) and monotonically increasing on $X \times Y$.*

Then $BN[G(1 \leftrightarrow 2)] \neq \emptyset$.

Proof. Another, and some times more convenient way of defining Nash equilibrium in the game Γ_B^* is via the best response correspondences that are defined as following. For all $j \in J_1$, $Br_j: \prod_{k \in J_2} \tilde{Y}_k \to 2^{\tilde{X}_j}$, $Br_j(\{\tilde{y}_k\}_{k \in J_2}) = \{\tilde{x}_j \in \tilde{X}_j : U_j(\tilde{x}_j, \{\tilde{y}_k\}_{k \in J_2}) \geq U_j(\tilde{x}_j', \{\tilde{y}_k\}_{k \in J_2}) \ \forall \ \tilde{x}_j' \in \tilde{X}_j\}$. For all $j \in J_2$, $Br_j: \prod_{k \in J_1} \tilde{X}_k \to 2^{\tilde{Y}_j}$, $Br_j(\{\tilde{x}_k\}_{k \in J_1}) = \{\tilde{y}_j \in \tilde{Y}_j : U_j(\{\tilde{x}_k\}_{k \in J_1}, \tilde{y}_j) \geq U_j(\{\tilde{x}_k\}_{k \in J_1}, \tilde{y}_j') \ \forall \tilde{y}_j' \in \tilde{Y}_j\}$. We define the following point-to-set mapping $Br(\{\tilde{x}_k\}_{k \in J_1}, \{\tilde{y}_k\}_{k \in J_2}) = \left(\{Br_j(\{\tilde{y}_k\}_{k \in J_2})\}_{j \in J_1}, \{Br_j(\{\tilde{x}_k\}_{k \in J_1})\}_{j \in J_2}\right)$, and if $\left(\{\tilde{x}_j^*\}_{l \in J_1}, \{\tilde{y}_k^*\}_{k \in J_2}\right) \in Br\left(\{\tilde{x}_j^*\}_{l \in J_1}, \{\tilde{y}_k^*\}_{k \in J_2}\right)$, then we have that $\left(\{\tilde{x}_j^*\}_{j \in J_1}, \{\tilde{y}_k^*\}_{k \in J_2}\right) \in NE[\Gamma_B^*]$. Here $\tilde{x}_j = \theta_1^j(y)$, $\forall j \in J_1$ and $\tilde{y}_k = \theta_2^k(x)$, $\forall k \in J_2$.

Denote by $\tilde{X} = \prod_{k \in J_1} \tilde{X}_k$ by $\tilde{Y} = \prod_{k \in J_2} \tilde{Y}_k$ and by grBr the graph of the point-to-set mapping Br. Since $\tilde{x}_j = \theta_1^j(y)$ $\forall j \in J_1$ and $\tilde{y}_k = \theta_2^k(x)$ $\forall k \in J_2$, then, according to condition 2), the sets $\tilde{X}_k$ $\forall k \in J_1$ and $\tilde{Y}_k$ $\forall k \in J_2$ are compacts and, according to the Tikhonov's theorem the sets $\tilde{X}$ and $\tilde{Y}$ are a non-empty compact and convex subsets of the Euclidean finite-dimensional space. For all $\{\tilde{y}_k\}_{k \in J_2}$ and $\{\tilde{x}_k\}_{k \in J_1}$ the sets $Br_j(\{\tilde{y}_k\}_{k \in J_2})$ and $Br_j(\{\tilde{x}_k\}_{k \in J_1})$ are non-empty because of to conditions 1) and 2). According to condition 3), $Br_j(\{\tilde{y}_k\}_{k \in J_2})$ and $Br_j(\{\tilde{x}_k\}_{k \in J_1})$ are also convex sets. Hence the set $Br(\{\tilde{x}_k\}_{k \in J_1}, \{\tilde{y}_k\}_{k \in J_2})$ is nonempty convex and compact for all $\tilde{x}_k \in \tilde{X}_k$, $k \in J_1$ and $\tilde{y}_k \in \tilde{Y}_k$, $k \in J_1$. According to condition 2) the mapping Br has a closed graph. Hence by Kakutani's theorem, the set-valued mapping Br has a fixed point. As we have noted, any fixed point is a Nash equilibrium. From the continuity of the functions $\theta_1^j, \forall j \in J_1$ and θ_2^k, $\forall k \in J_2$ it results that there exist inverse functions θ_1^j and θ_2^k for all $j = 1,2, k = 1,2$ and so, we have that for all $(\{\tilde{x}_l^*\}_{l \in J_1}, \{\tilde{y}_k^*\}_{k \in J_2}) \in NE[\Gamma_B^*]$ there is (x^*, y^*) such that $y^* = \vartheta_1^j(\tilde{x}_j^*)$ $\forall j \in J_1$ and $x^* = \vartheta_1^k(\tilde{y}_j^*)$ $\forall k \in J_2$. The theorem is completely proved.

■

In conclusion, we can build the following scheme for solving games of the type:

The game in *complete and perfect information* over the sets of pure strategies
$\Gamma = \langle \mathrm{I}; \mathrm{X}_\mathrm{i}, \mathrm{i} \in \mathrm{I}; \mathrm{H}_\mathrm{i}: \mathrm{X} \to \mathbb{R} \rangle$

⇓

Constract the game $Game(1 \leftrightarrow 2)$ *with incomplete and imperfect information* over the sets Θ_1 and Θ_2 of informational extended strategies.

⇓

Constract the incomplete and imperfect information game on the sets $\tilde{X}_j$ and $\tilde{Y}_k$ of informational non extended strategies generated by the informational extended strategies of the player 1 and 2 respectively.

⇓

Construct the Bayesian game $\Gamma_B = \left\langle I = \{1,2\}, S_1(\Delta_1), S_2(\Delta_2), \Delta_1, \Delta_2, p, q, \widetilde{H}_1, \widetilde{H}_1 \right\rangle$

⇓

Construct the Selten-Harsanyi game $\Gamma_\mathrm{B}^* = \left\langle \mathrm{J}, \{\mathrm{R}_\mathrm{j}\}_{\mathrm{j} \in \mathrm{J}}, \{\mathrm{U}_\mathrm{j}\}_{\mathrm{j} \in \mathrm{Jj}} \right\rangle$ with complete and imperfect information on the sets of the non-informational extended strategies

⇓

Determine Nash equilibrium profiles in the game Γ_B^* that is the Bayes-Nash equilibrium in the game Γ_B

⇓

Constract the non informational extended strategy profile $(\mathrm{x}^*, \mathrm{y}^*)$ which is generated by the Nash strategy profile in the game Γ_B^*

The following example illustrate the above context.

Example 5.1. *Consider the two persons game in the complete and perfect information, for which* $X = [0,1]$, $Y = [0,1]$, *are the sets of strategies and* $H_1(x, y) = \frac{3}{2}xy - x^2$, $H_2(x, y) = \frac{3}{2}xy - y^2$ *the payoff functions of the player. Solve this game using the described above Harsanyi's principle*.

Solution. In the capacity of the informational extended strategies we will use the $\theta_1^1(y) = \underset{x \in X}{\operatorname{argmax}} H_1(x, y) = \frac{3}{4}y$, $\theta_1^2(y) = y^2$, $\theta_2^1(x) = \underset{y \in Y}{\operatorname{argmax}} H_2(x, y) = \frac{3}{4}x$, $\theta_2^2(x) = x^2$. If players will use these strategies then the payoff functions will be $H_1(x,y)=\frac{3}{2}\left(\frac{3}{4}y\right)\underbrace{\left(\frac{3}{4}x\right)}-\left(\frac{3}{4}y\right)^2=\frac{27}{32}xy-\frac{9}{16}y^2$ and $H_2(x,y)=\frac{3}{2}\underbrace{\left(\frac{3}{4}y\right)}\left(\frac{3}{4}x\right)-\left(\frac{3}{4}x\right)^2=\frac{27}{32}xy-\frac{9}{16}x^2$. But because the player 1, for example, will not know that the player 2 as information extended strategy choose exactly $\theta_2(x)=\frac{3}{4}x$, then he will not know its payoff function. So the informational extended strategies generate uncertainty of the payoff functions, that we already have a incomplete information game.

Construct the Bayesian game Γ_B associated to the initial informational extended game.

a) The actions sets are $X=[0,1]$ and $Y=[0,1]$.

b) The set of the type for player 1 is $\Delta_1=\{\delta_1^1,\delta_1^2\}$ and for player 2 is $\Delta_2=\{\delta_2^1,\delta_2^2\}$ that means the following: player 1 (respectively 2) is of δ_1^1 type (respectively δ_2^1) if he choose the informational extended strategy $\theta_1^1(y)=\frac{3}{4}y$ (respectively $\theta_2^1(x)=\frac{3}{4}x$) and of the δ_1^2 type (respectively δ_2^2) if he choose the informational extended strategy $\theta_1^2(y)=y^2$ (respectively $\theta_2^2(x)=x^2$). Type-players will be denoted by $J_1=\{1,2\}, J_2=\{1,2\}$.

c) If the player 2 is of the type δ_2^k, then he supposes with the probability $p\left(\delta_1^j/\delta_2^k\right)=p(\delta_1^j)=\begin{cases}p\ for\ j=1\\1-p\ for\ j=2\end{cases}$ that the player 1 is of the type δ_1^j and, respectively, if the player 1 is of the type δ_1^j, then he suppose with the probability $p(\delta_2^k/\delta_1^j)=p(\delta_2^k)=\begin{cases}q\ for\ k=1\\1-q\ for\ k=2\end{cases}$ that the player 2 is of the type δ_2^k.

d) The strategies sets of the players are the following. $S_1(\delta_1^1)\equiv\tilde{X}_1=\left\{\tilde{x}_1\in[0,1]:\tilde{x}_1=\frac{3}{4}y,\forall y\in[0,1]\right\}=\left[0,\frac{3}{4}\right]\subseteq X$, $S_1(\delta_1^2)\equiv\tilde{X}_2=\{\tilde{x}_2\in[0,1]:\tilde{x}_2=y^2,\forall y\in[0,1]\}=[0,1]$, $S_2(\delta_2^1)\equiv\tilde{Y}_1=\left\{\tilde{y}_1\in[0,1]:\tilde{y}_1=\frac{3}{4}x,\forall x\in[0,1]\right\}=\left[0,\frac{3}{4}\right]\subseteq Y$, $S_2(\delta_2^2)\equiv\tilde{Y}_2=\{\tilde{y}_2\in[0,1]:\tilde{y}_2=x^2,\forall x\in[0,1]\}=[0,1]$.

e) The payoff functions of the player 1 is $\widetilde{H}_1\left(\tilde{x}_j,\tilde{y}_k,\delta_1^j,\delta_1^k\right)\equiv H_1\left(\tilde{x}_j,\tilde{y}_k\right)$ and of the player 2 is $\widetilde{H}_2\left(\tilde{x}_j,\tilde{y}_k,\delta_1^j,\delta_1^k\right)\equiv H_2\left(\theta_1^j(y),\theta_2^k(x)\right)$. Finally

$$\widetilde{H}_1\left(\tilde{x}_j, \tilde{y}_k, \delta_1^j, \delta_1^k\right) = \begin{cases} \frac{3}{2}\tilde{x}_1\tilde{y}_1 - (\tilde{x}_1)^2, & j = 1, k = 1, \\ \frac{3}{2}\tilde{x}_2\tilde{y}_1 - (\tilde{x}_2)^2, & j = 2, k = 1, \\ \frac{3}{2}\tilde{x}_1\tilde{y}_2 - (\tilde{x}_1)^2, & j = 1, k = 2, \\ \frac{3}{2}\tilde{x}_2\tilde{y}_2 - (\tilde{x}_2)^2, & j = 2, k = 2, \end{cases}$$

and

$$\widetilde{H}_2\left(\tilde{x}_j, \tilde{y}_k, \delta_1^j, \delta_1^k\right) = \begin{cases} \frac{3}{2}\tilde{x}_1\tilde{y}_1 - (\tilde{y}_1)^2, & j = 1, k = 1, \\ \frac{3}{2}\tilde{x}_2\tilde{y}_1 - (\tilde{y}_2)^2, & j = 2, k = 1, \\ \frac{3}{2}\tilde{x}_1\tilde{y}_2 - (\tilde{y}_1)^2, & j = 1, k = 2, \\ \frac{3}{2}\tilde{x}_2\tilde{y}_2 - (\tilde{y}_2)^2, & j = 2, k = 2. \end{cases}$$

Thus the Bayesian game is $\Gamma_B = \left\langle I = \{1,2\}, \mathcal{S}_1(\Delta_1), \mathcal{S}_2(\Delta_2), \Delta_1, \Delta_2, p, q, \widetilde{H}_1, \widetilde{H}_2 \right\rangle$.

Now we can construct the game Γ_B^* in complete and imperfect informations, associated to the Bayesian game recently constructed. The set of the type-players is $J = J_1 \cup J_2$, where $J_1 = \left\{j = (1, \theta_1^j) | j = 1,2\right\} = \{1,2\}$ and $J_2 = \{k = (2, \theta_2^k) | k = 1,2\} = \{3,4\}$. Finally, $J = \{1,2,3,4\}$. The set of the strategy of the type-player $j \in J$ *is* $R_j = \begin{cases} \tilde{X}_j & j \in J_1, \\ \tilde{Y}_j & j \in J_2. \end{cases}$ and the strategy of the type-player $j \in J$ is $r_j = \begin{cases} \tilde{x}_j \in \tilde{X}_j & j \in J_1, \\ \tilde{y} \in \tilde{Y}_j & j \in J_2 \end{cases}$. So, for type player $j = 1$ the strategy set is $R_1 = \tilde{X}_1 = \left[0, \frac{3}{4}\right]$, for type-player $j = 2$ the strategy set is $R_2 = \tilde{X}_2 = [0,1]$, for type-player $j = 3$ (or $k = 1$) $R_3 = \tilde{Y}_1 = \left[0, \frac{3}{4}\right]$ and for type-player $j = 4$, the strategy set is $R_4 = \tilde{Y}_2 = [0,1]$. According to these, the strategy profile in the game Γ_B^* is $r = (r_1, r_2, r_3, r_4) = (\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2)$, where $\tilde{x}_1 \in \left[0, \frac{3}{4}\right]$, $\tilde{x}_2 \in [0,1]$, $\tilde{y}_1 \in \left[0, \frac{3}{4}\right]$ and $\tilde{y}_2 \in [0,1]$. Payoff functions of the type-players are defined as following

$$\mathrm{U}_1(\tilde{x}_1, \tilde{y}_1, \tilde{y}_2, q) = qH_1(\tilde{x}_1, \tilde{y}_1) + (1 - q)H_1(\tilde{x}_1, \tilde{y}_2) == -(\tilde{x}_1)^2 + \frac{3}{2}\tilde{x}_1(q\tilde{y}_1 + (1 - q)\tilde{y}_2),$$

$$\mathrm{U}_2(\tilde{x}_2, \tilde{y}_1, \tilde{y}_2, q) = qH_1(\tilde{x}_2, \tilde{y}_1) + (1 - q)H_1(\tilde{x}_2, \tilde{y}_2) == -(\tilde{x}_2)^2 + \frac{3}{2}\tilde{x}_2(q\tilde{y}_1 + (1 - q)\tilde{y}_2),$$

$$U_3(\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, p) = pH_2(\tilde{x}_1, \tilde{y}_1) + (1 - p)H_2(\tilde{x}_2, \tilde{y}_1) = -(\tilde{y}_1)^2 + \frac{3}{2}\tilde{y}_1(p\tilde{x}_1 + +(1 - p)\tilde{x}_2),$$

$$U_4(\tilde{x}_1, \tilde{x}_2, \tilde{y}_2, p) = pH_2(\tilde{x}_1, \tilde{y}_2) + (1 - p)H_2(\tilde{x}_2, \tilde{y}_2) == -(\tilde{y}_2)^2 + \frac{3}{2}\tilde{y}_2(p\tilde{x}_1 + (1 - p)\tilde{x}_2).$$

Thus we have obtained the following strategic game

$$\Gamma_B^* = \langle J = \{1,2,3,4\}, R_j, U_j \rangle.$$

Now, we can determine the equilibrium profile. Strategy profiles $(\tilde{x}_1^*, \tilde{x}_2^*, \tilde{y}_1^*, \tilde{y}_2^*) \in NE(\Gamma_B^*)$ using the "best response approach

$$\begin{cases} \tilde{x}_1^* \in Br_1(\tilde{y}_1^*, \tilde{y}_2^*, q), \\ \tilde{x}_2^* \in Br_2(\tilde{y}_1^*, \tilde{y}_2^*, q), \\ \tilde{y}_1^* \in Br_3(\tilde{x}_1^*, \tilde{x}_2^*, p), \\ \tilde{y}_2^* \in Br_4(\tilde{x}_1^*, \tilde{x}_2^*, p), \end{cases}$$

Where

$$Br_1(\tilde{y}_1^*, \tilde{y}_2^*, q) = Arg \max_{\tilde{x}_1 \in \tilde{X}_1} U_1(\tilde{x}_1, \tilde{y}_1^*, \tilde{y}_2^*, q),$$

$$Br_2(\tilde{y}_1^*, \tilde{y}_2^*, q) = Arg \max_{\tilde{x}_2 \in \tilde{X}_2} U_2(\tilde{x}_2, \tilde{y}_1^*, \tilde{y}_2^*, q),$$

$$Br_3(\tilde{x}_1^*, \tilde{x}_2^*, p) = Arg \max_{\tilde{y}_1 \in \tilde{Y}_1} U_3(\tilde{x}_1^*, \tilde{x}_2^*, \tilde{y}_1, p),$$

$$Br_4(\tilde{x}_1^*, \tilde{x}_2^*, p) = Arg \max_{\tilde{y}_2 \in \tilde{Y}_2} U_4(\tilde{x}_1^*, \tilde{x}_2^*, \tilde{y}_2, p).$$

This is equivalent to the following system

$$\begin{cases} \tilde{x}_1^* = \frac{3}{4}[q\tilde{y}_1^* + (1-q)\tilde{y}_2^*] \in \left[0, \frac{3}{4}\right], \\ \tilde{x}_2^* = \frac{3}{4}[q\tilde{y}_1^* + (1-q)\tilde{y}_2^*] \in [0,1], \\ \tilde{y}_1^* = \frac{3}{4}[p\tilde{x}_1^* + (1-p)\tilde{x}_2^*] \in \left[0, \frac{3}{4}\right], \\ \tilde{y}_2^* = \frac{3}{4}[p\tilde{x}_1^* + (1-p)\tilde{x}_2^*] \in 0,1]. \end{cases}$$

In the particular case, if the player 1 chooses the informational extended strategy $\theta_1^1(y) = \frac{3}{4}y$ and assume with probability q that player 2 chooses the informational extended strategy $\theta_2^1(x) = \frac{3}{4}x$, and with probability $1-q$ the informational extended strategy $\theta_2^2(x) = x^2$, and, respectively, if the player 2 chooses the informational extended strategy $\theta_2^1(x) = \frac{3}{4}x$ and assume with probability p that player 1 chooses the informational extended strategy $\theta_1^1(y) = \frac{3}{4}y$ and with probability $1-p$ the informational extended strategy $\theta_1^2(y) = y^2$, then the informational non extended Bayes-Nash equilibrium profiles $(x^*(q), y^*(p))$ is calculated from the following system

$$\begin{cases} \theta_1^1(y) = \frac{3}{4}[q\theta_2^1(x) + (1-q)\theta_2^2(x)], \\ \theta_2^1(x) = \frac{3}{4}[p\theta_1^1(y) + (1-p)\theta_1^2(y)]. \end{cases}$$

So we have the system

$$\begin{cases} \frac{3}{4}y = \frac{3}{4}\left[q\frac{3}{4}x + (1-q)x^2\right], \\ \frac{3}{4}x = \frac{3}{4}\left[p\frac{3}{4}y + (1-p)\frac{3}{4}x\right]. \end{cases}$$

Finally, all solutions of this example are described in the following table.

Players type	Informational extended strategies	Solutions from system
$(1,1) == [(1,\theta_1^1),(2,\theta_2^1)]$	$\left(\theta_1^1(y),\theta_2^1(y)\right) = = \left(\frac{3}{4}y,\frac{3}{4}x\right)$	$\begin{cases} y = \left[q\frac{3}{4}x + (1-q)x^2\right] \\ x = \left[p\frac{3}{4}y + (1-p)\frac{3}{4}x\right] \end{cases}$
$(2,1) == [(2,\theta_1^2),(2,\theta_2^1)]$	$\left(\theta_1^2(y),\theta_2^1(y)\right) = = \left(y^2,\frac{3}{4}x\right)$	$\begin{cases} y^2 = \frac{3}{4}\left[q\frac{3}{4}x + (1-q)x^2\right] \\ x = \left[p\frac{3}{4}y + (1-p)\frac{3}{4}x\right] \end{cases}$
$(1,2) == [(1,\theta_1^1),(2,\theta_2^2)]$	$\left(\theta_1^1(y),\theta_2^2(y)\right) = = \left(\frac{3}{4}y,x^2\right)$	$\begin{cases} y = \left[q\frac{3}{4}x + (1-q)x^2\right] \\ x = \left[p\frac{3}{4}y + (1-p)\frac{3}{4}x\right] \end{cases}$
$(2,2) == [(2,\theta_1^2),(2,\theta_2^2)]$	$\left(\theta_1^2(y),\theta_2^2(y)\right) == (y^2,x^2)$	$\begin{cases} y^2 = \frac{3}{4}\left[q\frac{3}{4}x + (1-q)x^2\right] \\ x^2 = \frac{3}{4}\left[p\frac{3}{4}y + (1-p)y^2)\right] \end{cases}$

With this we finished solving the example.

6. Approaches for solving bimatrix informational extended games

6.1 Bimtrix informational extended games

We consider the informational non extended bimatrix game in the strategic form

$$\Gamma = \langle I, J, A, B \rangle, \qquad (6.1.1)$$

where $I = \{1,2,..,n\}$ is the line index set (the set of strategies of the player 1), $J = \{1,2,..,m\}$ is the column index set (the set of strategies of the player 2) and $A = ||a_{ij}||_{i\in I}^{j\in J}$, $B = ||b_{ij}||_{i\in I}^{j\in J}$ are the payoff matrices of player 1 and player 2, respectively. All players know exactly the payoff matrices and the sets of strategies. Players maximize their payoffs. So the game is in **complete information** (the players know exactly the normal form of the game). As will mention above, we assign to players an additional characteristic which we call *an informational type of the payer*. More exactly , we say that the player 1 is of the "2 → 1 informational type" and respectively, the player 2 is of the "1 → 2 informational type" if the player 1 (respectively player 2) knows the precise value of the strategy which will be chosen by the player 2 (respectively by the player 1). These conditions stipulate that we can analyze the informational extension of the game generated by a double-sided informational flow, denoted by $1 \rightleftarrows 2$. It means the player 1 knows exactly the value of the strategy chosen by the player 2, as well as, simultaneously, the player 2 knows exactly the value of the strategy chosen by the player 1. So the game (6.1.1) is in **perfect information** over the sets of pure strategies.

The conditions described above stipulate that we can use the set of informational extended strategies of the player 1 (respectively 2) which is the set of the functions $\Theta_1 = \{\theta_1^\alpha : J \to I\}_{\alpha=1}^{\chi_1}$ and, respectively $\Theta_2 = \left\{\theta_2^\beta : I \to J\right\}_{\beta=1}^{\chi_2}$. It is easy to see that $\chi_1 = n^m$ and $\chi_2 = m^n$. Thus, the informational extended strategies of the player 1 are the functions θ_1^α such that, for all $j \in J$, there is $i_j^\alpha \in I$ such that $\theta_1^\alpha(j) = i_j^\alpha$ and it means the following: the player 1 will choose the line $i_j^\alpha \in I$ if the player 2 will choose the column $j \in J$. Respectively, the informational extended strategies of the player 2 are functions θ_2^β such that, for all $i \in I$, there is $j_i^\beta \in J$ such that $\theta_2^\beta(i) = j_i^\beta$ and it means the following: the player 2 will choose the column $j_i^\beta \in J$ if the player 1 will choose the line $i \in I$.

It should be mentioned that the players do not know the informational type of each other. In other words, the players do not know the informational extended strategies of each others and from this point of view we can consider that the game is in **imperfect information** structure over the sets of the informational extended strategies.

Denote by $Game(1 \rightleftarrows 2)$ the bimatrix game in the informational extended strategies, described above. Remark that the notation $Game(1 \rightleftarrows 2)$ does not represent the normal form. This game is in imperfect information on the set of informational extended strategies, but because we do not know yet the normal form, we can not say if this game is in complete or incomplete information. The quantification of information in the games of type $Game(1 \leftrightarrow 2)$ is done by means of functions which represent informational extended strategies. As mentioned above, we can use the following approach to solve the informational extended game $Game(1 \leftrightarrow 2)$.

6.2 Solving the bimatrix informational extended game by means of the normal form.

Denote by

$$gr\theta_1^\alpha = \left\{(i,j) : j \in J, i \equiv i_j^\alpha = \theta_1^\alpha(j)\right\}, \quad gr\theta_2^\beta \left\{(i,j) : i \in I, j \equiv j_i^\beta = \theta_2^\beta(i)\right\}$$

the graphs of the informational extended strategies θ_1^α and θ_2^β. It is clear that $gr\theta_1^\alpha$ (respectively

$gr\theta_2^{\beta}$) is the set of the informational non extended strategy profiles generated by the informational extended strategy θ_1^{α} (respectively θ_2^{β}).

According to [Hancu 2011] we can construct the normal form of the informational extended game, denoted by

$$\Gamma(1 \rightleftarrows 2) = \langle I, \Theta_1, \Theta_2, A(1 \rightleftarrows 2), B(1 \rightleftarrows 2) \rangle$$

where the payoff matrices of the player 1 is $A(1 \rightleftarrows 2) = \left\| a_{\alpha\beta} \right\|_{\alpha=\overline{1,\chi_1}}^{\beta=\overline{1,\chi_2}}$, for

$$a_{\alpha\beta} = \begin{cases} \max\limits_{(i,j)\in\left[gr\theta_1^{\alpha} \cap gr\theta_2^{\beta}\right]} a_{ij} & if gr\theta_1^{\alpha} \cap gr\theta_2^{\beta} \neq \emptyset, \\ -\infty & if gr\theta_1 \cap gr\theta_2 = \emptyset, \end{cases} \qquad (6.2.2)$$

and of the player 2 is $B(1 \rightleftarrows 2) = \left\| b_{\alpha\beta} \right\|_{\alpha=\overline{1,\chi_1}}^{\beta=\overline{1,\chi_2}}$, for

$$b_{\alpha\beta} = \begin{cases} \max\limits_{(i,j)\in\left[gr\theta_1^{\alpha} \cap gr\theta_2^{\beta}\right]} b_{ij} & if gr\theta_1^{\alpha} \cap gr\theta_2^{\beta} \neq \emptyset, \\ -\infty & if gr\theta_1 \cap gr\theta_2 = \emptyset. \end{cases} \qquad (6.2.3)$$

The game $\Gamma(1 \rightleftarrows 2)$ is one in **complete** information because the players known exactly theirs payoff matrices and in **imperfect** information because the players do not know what kind of informational extended strategy will be chosen by each others.

Finally, to determine the Nash equilibrium profiles in the bimatrix informational extended game of type $\Gamma(1 \rightleftarrows 2)$ we have to do the following steps:

• construct the sets of the informational extended strategies of the players, i.e. $\Theta_1 = \{\theta_1^{\alpha}: J \to I\}_{\alpha=1}^{\chi_1}$ and $\Theta_2 = \left\{\theta_2^{\beta}: I \to J\right\}_{\beta=1}^{\chi_2}$;

• determine the sets of all non informational extended strategy profiles generated by the informational extended strategies θ_1^{α} and θ_2^{β}, i.e. $gr\theta_1^{\alpha}$, $gr\theta_2^{\beta}$ and intersection $gr\theta_1^{\alpha} \cap gr\theta_2^{\beta}$;

• construct the payoff matrices $A(1 \rightleftarrows 2)$ and $B(1 \rightleftarrows 2)$ according to the relations (6.2.2)-(6.2.3);

• using existent algorithms, to determine the Nash equilibrium profile in the bimatrix game with the matrices $A(1 \rightleftarrows 2)$ and $B(1 \rightleftarrows 2)$ from (6.2.2)-(6.2.3).

In the example 6.2.2 we have illustrated the described above methods.

Example 6.2.2 *We construct the normal form of the* $1 \leftrightarrow 2$ *informational extended games and determinate the Nash equilibrium profiles in the following bimatricial game* $H_1 = \begin{pmatrix} 3 & 5 & 4 \\ 6 & \mathbf{7} & 2 \end{pmatrix}$, $H_2 =$

$\begin{pmatrix} 0 & 5 & 1 \\ 4 & 3 & 2 \end{pmatrix}$.

Solution. It is easy to note that the Nash equilibrium profiles of the non informational extended game are $NE[\Gamma] = \{(2,1)\}$. We consider the action set strategies and the sets of the informational extended strategies $\Theta_1 = \left\{\theta_1^\alpha(j) = i_j, j = \overline{1,3}, i_j \in X_1^\alpha\right\}_{\alpha=\overline{1,8}}$, were $\theta_1^1(j) = 1\ \ \forall j = 1,2,3;\ \theta_1^2(j) = 2\ \ \forall j = 1,2,3;\ \theta_1^3(j) = \begin{cases} 1 \text{ if } j = 1,2, \\ 2 \text{ if } j = 3, \end{cases}$ $\theta_1^4(j) = \begin{cases} 1 \text{ if } j = 1,3, \\ 2 \text{ if } j = 2, \end{cases}$ $\theta_1^5(j) = \begin{cases} 1 \text{ if } j = 2,3, \\ 2 \text{ if } j = 1, \end{cases}$ $\theta_1^6(j) = \begin{cases} 1 \text{ if } j = 1, \\ 2 \text{ if } j = 2,3, \end{cases}$ $\theta_1^7(j) = \begin{cases} 1 \text{ if } j = 2, \\ 2 \text{ if } j = 1,3, \end{cases}$ $\theta_1^8(j) = \begin{cases} 1 \text{ if } j = 3, \\ 2 \text{ if } j = 1,2 \end{cases}$ of the players 1 and $\Theta_2 = \left\{\theta_2^\beta(i) = j_i, i = \overline{1,2}, j_i \in X_2^\beta\right\}_{\beta=\overline{1,9}}$ were $\theta_2^1(i) = 1\ \ \forall i = 1,2;\ \theta_2^2(i) = 2\ \ \forall i = 1,2;\ \theta_2^3(i) = 3\ \ \forall i = 1,2;\ \theta_2^4(i) = \begin{cases} 1 \text{ if } i = 1, \\ 2 \text{ if } i = 2, \end{cases}$ $\theta_2^5(i) = \begin{cases} 1 \text{ if } i = 2, \\ 2 \text{ if } i = 1, \end{cases}$ $\theta_2^6(i) = \begin{cases} 1 \text{ if } i = 1, \\ 3 \text{ if } i = 2, \end{cases}$ $\theta_2^7(i) = \begin{cases} 1 \text{ if } i = 2, \\ 3 \text{ if } i = 1, \end{cases}$ $\theta_2^8(i) = \begin{cases} 2 \text{ if } i = 1, \\ 3 \text{ if } i = 2, \end{cases}$ $\theta_2^9(i) = \begin{cases} 3 \text{ if } i = 1, \\ 2 \text{ if } i = 2. \end{cases}$ To determine payoffs of the players in the following table we represent the graph intersections $gr\theta_1^\alpha \cap gr\theta_2^\beta$ for $\alpha = \overline{1,8}$ and $\beta = \overline{1,9}$:

$\cap$	$gr\theta_2^1$	$gr\theta_2^2$	$gr\theta_2^3$	$gr\theta_2^4$	$gr\theta_2^5$	$gr\theta_2^6$	$gr\theta_2^7$	$gr\theta_2^8$	$gr\theta_2^9$
$gr\theta_1^1$	(1,1)	(1,2)	(1,3)	(1,1)	(1,2)	(1,1)	(1,3)	(1,2)	(1,3)
$gr\theta_1^2$	(2,1)	(2,2)	(2,3)	(2,2)	(2,1)	(2,3)	(2,1)	(2,3)	(2,2)
$gr\theta_1^3$	(1,1)	(1,2)	(2,3)	(1,1)	(1,2)	(1,1) (2,3)	$\emptyset$	(1,2) (2,3)	$\emptyset$
$gr\theta_1^4$	(1,1)	(2,2)	(1,3)	(1,1) (2,2)	$\emptyset$	(1,1)	(1,3)	$\emptyset$	(1,3)
$gr\theta_1^5$	(2,1)	(1,2)	(1,3)	$\emptyset$	(1,2) (2,1)	$\emptyset$	(1,1)	(1,2)	(1,3)
$gr\theta_1^6$	(1,1)	(2,2)	(2,3)	(1,1) (2,2)	$\emptyset$	(1,1) (2,3)	$\emptyset$	(2,2)	(2,2)
$gr\theta_1^7$	(2,1)	(1,2)	(2,3)	$\emptyset$	(1,2) (2,1)	(2,3)	(2,1)	(1,2) (2,3)	$\emptyset$
$gr\theta_1^8$	(2,1)	(2,2)	(1,3)	(2,2)	(2,1)	$\emptyset$	(1,3) (2,1)	$\emptyset$	(1,3) (2,2)

Using this table we can construct the normal form of the $1 \leftrightarrow 2$ informational extended bimatricial game with the following payoff matrices for the player 1

$$A(1 \rightleftarrows 2) = \begin{pmatrix} 3\ 5\ 4 & 3 & 5 & 3 & 4 & 5 & 4 \\ 6\ 7\ 2 & 7 & 6 & 2 & 6 & 2 & 7 \\ 3\ 5\ 2 & 3 & 5 & 3 & -\infty & 3 & -\infty \\ 3\ 7\ 4 & 7 & -\infty & 3 & 4 & -\infty & 7 \\ 6\ 5\ 4 & -\infty & 7 & -\infty & 3 & 5 & 4 \\ 3\ 7\ 2 & 7 & -\infty & 3 & -\infty & 2 & 7 \\ 6\ 7\ 4 & -\infty & 6 & 2 & 6 & 5 & -\infty \\ 6\ 7\ 4 & 7 & 6 & -\infty & 6 & -\infty & 7 \end{pmatrix}$$

and for the player 2 correspondingly

$$B(1 \rightleftarrows 2) = \begin{pmatrix} 0\ 5\ 1 & 0 & 5 & 0 & 1 & 5 & 1 \\ 4\ 3\ 2 & 3 & 4 & 2 & 4 & 2 & 3 \\ 0\ 5\ 2 & 0 & 5 & 2 & -\infty & 5 & -\infty \\ 0\ 3\ 1 & 3 & -\infty & 0 & 1 & -\infty & 3 \\ 4\ 5\ 1 & -\infty & 5 & -\infty & 0 & 5 & 1 \\ 0\ 3\ 2 & 3 & -\infty & 2 & -\infty & 2 & 3 \\ 4\ 5\ 2 & -\infty & 5 & 2 & 4 & 5 & -\infty \\ 4\ 3\ 1 & 3 & 4 & -\infty & 4 & -\infty & 3 \end{pmatrix}.$$

In the table below is shown the accordance between Nash equilibrium profiles in the $\Gamma(1 \leftrightarrow 2)$ game and profiles in the non informational extended game Γ:

$\Gamma(1 \leftrightarrow 2)$	(θ_1^1, θ_2^8)	(θ_1^2, θ_2^1)	(θ_1^2, θ_2^7)	(θ_1^4, θ_2^2)	(θ_1^4, θ_2^4)	(θ_1^4, θ_2^9)	(θ_1^5, θ_2^5)	(θ_1^5, θ_2^8)
Γ	(1,2)	(2,1)	(2,1)	(2,2)	(2,2)	(2,2)	(2,1)	(1,2)
$\Gamma(1 \leftrightarrow 2)$	(θ_1^6, θ_2^2)	(θ_1^6, θ_2^4)	(θ_1^6, θ_2^9)	(θ_1^7, θ_2^2)	(θ_1^7, θ_2^8)	(θ_1^8, θ_2^1)	(θ_1^8, θ_2^7)	
Γ	(2,2)	(2,1)	(2,2)	(1,2)	(1,2)	(2,1)	(2,1)	

6.3 Solving the bimatrix informational extended game by means of the informational non extended game

As described above we can describe the informational extended strategies in bimatrix game as follows: to all informational extended strategies θ_1^α, respectively θ_2^β, we put in correspondence a set

$$I^\alpha = \{i_j^\alpha : i_j^\alpha \in I, \forall j = \overline{1,m}\} \text{ and } J^\beta = \{j_i^\beta : j_i^\beta \in J, \forall i = \overline{1,n}\}.$$

So, for all $j \in J$, $\theta_1^\alpha(j) = i_j^\alpha \in I^\alpha$ and for all $i \in I$, $\theta_2^\beta(i) = j_i^\beta \in J^\beta$. Denote by $gr\theta_1^\alpha = \{(j, i_j^\alpha) \equiv (i_j^\alpha, j) : j \in J, i_j^\alpha \in I^\alpha\}$ and $gr\theta_2^\beta = \{(i, j_i^\beta) \equiv (j_i^\beta, i) : i \in I, j_i^\beta \in J^\beta\}$ the sets of the informational non extended strategy profiles of the player 1, respectively 2, generated by the informational extended strategies θ_1^α and θ_2^β, respectively. Denote by

$$difI^\alpha = \{i_j^\alpha \in I^\alpha : i_j^\alpha \neq i_k^\alpha, \forall j, k \in J, j \neq k\}$$

and

$$difJ^{\beta} = \left\{j_i^{\beta} \in J^{\beta}: j_i^{\beta} \neq j_r^{\beta} \forall i, r \in I, i \neq r\right\}.$$

Then the set $difI^{\alpha}$, respectively $difJ^{\beta}$, is the set of informational non extended strategies of the player 1, respectively 2, generated by the informational extended strategies θ_1^{α}, respectively θ_2^{β}. Here $\alpha = \overline{1, n^m}$ and $\beta = \overline{1, m^n}$. Using these notations, we can represent the informational extended strategies θ_1^{α}, respectively θ_2^{β}, by the cortege $\mathcal{I}^{\alpha} = \left(i_1^{\alpha}, i_2^{\alpha}, \dots, i_j^{\alpha}, \dots i_m^{\alpha}\right)$ where: $i_j^{\alpha} \in I^{\alpha}, \forall j = \overline{1, m}$, respectively $\mathcal{J}^{\beta} = \left(j_1^{\beta}, j_2^{\beta}, \dots, j_i^{\beta}, \dots j_n^{\beta}\right)$, where $j_i^{\beta} \in J^{\beta}, \forall i = \overline{1, n}$.

Now, according to [Hancu 2012], for all fixed informational extended strategy profile $\theta_1^{\alpha}, \theta_2^{\beta}$, we can construct the normal form of the bimatrix game

$$\Gamma\left(\theta_1^{\alpha}, \theta_2^{\beta}\right) = \langle I, J, A^{\alpha}, B^{\beta} \rangle, \qquad (6.3.4)$$

that is an informational non extended game generated by the informational extended strategies $\left(\theta_1^{\alpha}, \theta_2^{\beta}\right)$. Here $A^{\alpha} = ||a_{i_j^{\alpha} j_i^{\beta}}||_{i \in I}^{j \in J}$, $B^{\beta} = ||b_{i_j^{\alpha} j_i^{\beta}}||_{i \in I}^{j \in J}$, $i_j^{\alpha} \in \mathcal{I}^{\alpha}$, $j_i^{\beta} \in \mathcal{J}^{\beta}$. The game $\Gamma\left(\theta_1^{\alpha}, \theta_2^{\beta}\right)$ is played as follows: independently and simultaneously each player $k = \overline{1,2}$ chooses the informational non extended strategy $i \in I$, $j \in J$ after that players 1 and 2 calculate the value of the informational extended strategies $i_j^{\alpha} = \theta_1^{\alpha}(j) \in \mathrm{I}$ and $j_i^{\beta} = \theta_2^{\beta}(i) \in \mathrm{J}$, and further each player calculates the payoff values $a_{i_j^{\alpha} j_i^{\beta}}$, $b_{i_j^{\alpha} j_i^{\beta}}$, and with this the game is finished. It is clear that for all strategy profiles (i, j) in the game $\Gamma = \langle I, J, A, B \rangle$ from (6.1.1) the following realization $\left(i_j^{\alpha} = \theta_1^{\alpha}(j), j_i^{\beta} = \theta_2^{\beta}(i)\right)$ in terms of the informational extended strategies will correspond. The game (6.3.4) is the bimatrix game with complete and imperfect information over the set of informational non extended strategies I, J.

Finally, to determine the Nash equilibrium profiles in the bimatrix game of type $\Gamma\left(\theta_1^{\alpha}, \theta_2^{\beta}\right)$ defined in (6.3.4) we have to the following steps:

- using the "combinatorial algorithm" construct the corteges $\mathcal{I}^{\alpha}, \mathcal{J}^{\beta}$, for all α, β;
- for all fixed α, β, construct the payoff matrices $A^{\alpha} = ||a_{i_j^{\alpha} j_i^{\beta}}||_{i \in I}^{j \in J}$, $B^{\beta} = ||b_{i_j^{\alpha} j_i^{\beta}}||_{i \in I}^{j \in J}$;
- using existent algorithms determine the set $NE\left(A^{\alpha}, B^{\beta}\right)$ of Nash equilibrium profiles in the bimatrix game with the matrices A^{α} and B^{β}.

We illustrate the described above method in the following example:

Example 6.3.1 *We consider the bimatrix game* $H_1 = \begin{pmatrix} 3 & 5 & 4 \\ 6 & 7 & 2 \end{pmatrix}$, $H_2 = \begin{pmatrix} 0 & 5 & 1 \\ 4 & 3 & 2 \end{pmatrix}$ *and construct the normal form of the game generated by the informational extended strategies.*

Solution. Consider the sets of the informational extended strategies from the Example 6.2.2. The corteges $\mathcal{I}^{\alpha}$ and $\mathcal{J}^{\beta}$ are:

- for player 1: $\theta_1^1 \Rightarrow \mathcal{I}^1 = (1,1,1)$; $\theta_1^2 \Rightarrow \mathcal{I}^2 = (2,2,2)$; $\theta_1^3 \Rightarrow \mathcal{I}^3 = (1,1,2)$; $\theta_1^4 \Rightarrow \mathcal{I}^4 = (1,2,1)$; $\theta_1^5 \Rightarrow \mathcal{I}^5 = (2,1,1)$; $\theta_1^6 \Rightarrow \mathcal{I}^6 = (1,2,2)$; $\theta_1^7 \Rightarrow \mathcal{I}^7 = (2,1,2)$; $\theta_1^8 \Rightarrow \mathcal{I}^8 = (2,2,1)$;

- for player 2: $\theta_2^1 \Rightarrow \mathcal{J}^1 = (1,1)$; $\theta_2^2 \Rightarrow \mathcal{J}^2 = (2,2)$; $\theta_2^3 \Rightarrow \mathcal{J}^3 = (3,3)$; $\theta_2^4 \Rightarrow \mathcal{J}^4 = (1,2)$; $\theta_2^5 \Rightarrow \mathcal{J}^5 = (2,1)$; $\theta_2^6 \Rightarrow \mathcal{J}^6 = (1,3)$; $\theta_2^7 \Rightarrow \mathcal{J}^7 = (3,1)$; $\theta_2^8 \Rightarrow \mathcal{J}^8 = (2,3)$; $\theta_2^9 \Rightarrow \mathcal{J}^9 = (3,2)$.

So, we can construct the all amount, equal to 72, of the informational non extended game generated by all informational extended strategy profile $\left(\theta_1^{\alpha}, \theta_2^{\beta}\right)$:

$$\Gamma(\theta_1^1, \theta_2^1) = \left\langle I, J, A^1 = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix}, B^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle;$$ [4]

$$\Gamma(\theta_1^1, \theta_2^2) = \left\langle I, J, A^1 = \begin{pmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \end{pmatrix}, B^2 = \begin{pmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \end{pmatrix} \right\rangle;$$

$$\Gamma(\theta_1^1, \theta_2^3) = \left\langle I, J, A^1 = \begin{pmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \end{pmatrix}, B^3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \right\rangle;$$

$$\Gamma(\theta_1^3, \theta_2^4) = \left\langle I, J, A^3 = \begin{pmatrix} 3 & 3 & 6 \\ 5 & 5 & 7 \end{pmatrix}, B^4 = \begin{pmatrix} 0 & 0 & 4 \\ 5 & 5 & 3 \end{pmatrix} \right\rangle;$$ [5]

$$\Gamma(\theta_1^3, \theta_2^5) = \left\langle I, J, A^3 = \begin{pmatrix} 5 & 5 & 7 \\ 3 & 3 & 6 \end{pmatrix}, B^5 = \begin{pmatrix} 5 & 5 & 3 \\ 0 & 0 & 4 \end{pmatrix} \right\rangle;$$

$$\Gamma(\theta_1^4, \theta_2^4) = \left\langle I, J, A^4 = \begin{pmatrix} 3 & 6 & 3 \\ 5 & 7 & 5 \end{pmatrix}, B^4 = \begin{pmatrix} 0 & 4 & 0 \\ 5 & 3 & 5 \end{pmatrix} \right\rangle;$$

$$\Gamma(\theta_1^4, \theta_2^5) = \left\langle I, J, A^4 = \begin{pmatrix} 5 & 7 & 5 \\ 3 & 6 & 3 \end{pmatrix}, B^5 = \begin{pmatrix} 5 & 3 & 5 \\ 0 & 4 & 0 \end{pmatrix} \right\rangle;$$

$$\vdots$$

$$\Gamma(\theta_1^8, \theta_2^8) = \left\langle I, J, A^8 = \begin{pmatrix} 7 & 7 & 5 \\ 2 & 2 & 4 \end{pmatrix}, B^8 = \begin{pmatrix} 3 & 3 & 5 \\ 2 & 2 & 1 \end{pmatrix} \right\rangle;$$

$$\Gamma(\theta_1^8, \theta_2^9) = \left\langle I, J, A^8 = \begin{pmatrix} 2 & 2 & 4 \\ 7 & 7 & 5 \end{pmatrix}, B^9 = \begin{pmatrix} 2 & 2 & 1 \\ 3 & 3 & 5 \end{pmatrix} \right\rangle.$$

6.4 Bayes-Nash solutions in the bimatrix informational extended games

As was mentioned in above any strategy profile $(\theta_1^{\alpha}, \theta_2^{\beta})$ in informational extended strategies generates a couple of matrices, which represent the utility of the players in informational non extended

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[4] We observe that for any i=1,2 and j=1,2,3 the value of information extended strategy $\theta_1^1 = 1, \theta_2^1 = 1$. So $a_{i_j^{\alpha} j_i^{\beta}} = 1, b_{i_j^{\alpha} j_i^{\beta}} = 1$

[5] We determine elemental (I,j)=(1,3) of the matrices from $\Gamma(\theta_1^3, \theta_2^4)$. We observe that for $\theta_1^3 \Rightarrow (1,1,2), \theta_2^4 \Rightarrow (1,2)$. So $i_j^3 = 2, j_i^4 = 1$ si $a_{i_j^3 j_i^4} = 6, b_{i_j^3 j_i^4} = 4$

strategies

$$\left\{A(\alpha,\beta) = ||a_{i_j^\alpha j_i^\beta}||_{i\in I}^{j\in J}, B(\alpha,\beta) = ||b_{i_j^\alpha j_i^\beta}||_{i\in I}^{j\in J}, i_j^\alpha \in I^\alpha, j_i^\beta \in J^\beta\right\}_{\alpha=\overline{1,\kappa_1}}^{\beta=\overline{1,\kappa_2}}.$$

So as the players do not know what informational extended strategies are chosen by their partners, each player will have a possible set of utility matrices. This type of games is one in **incomplete information** because neither player 1 nor player 2 knows exactly which matrix from the mentioned set of matrices will be his utility.

Finally, the game $Game(1 \leftrightarrow 2)$ of imperfect information on the set of informational extended strategies, generates an incomplete information game on the set of informational non extended strategies. So we study the following two person game: the strategies of the player 1 are $I = \{1,2,\dots,n\}$ and of the player 2 are $J = \{1,2,\dots,m\}$; the payoff matrix of the player 1 is one of the matrices from the set $\left\{A(\alpha,\beta) = ||a_{i_j^\alpha j_i^\beta}||_{i\in I}^{j\in J}, i_j^\alpha \in I^\alpha, j_i^\beta \in J^\beta\right\}_{\alpha=\overline{1,\chi_1}}^{\beta=\overline{1,\chi_2}}$ and the payoff matrix of the player 2 is one of the matrices from the set $\left\{B(\alpha,\beta) = ||b_{i_j^\alpha j_i^\beta}||_{i\in I}^{j\in J}, i_j^\alpha \in I^\alpha, j_i^\beta \in J^\beta\right\}_{\alpha=\overline{1,\kappa_1}}^{\beta=\overline{1,\kappa_2}}$.

When, using the informational extended strategies, the matrices $A\left(\theta_1^\alpha, \theta_2^\beta\right) \equiv A(\alpha,\beta)$ and $B\left(\theta_1^\alpha, \theta_2^\beta\right) \equiv B(\alpha,\beta)$ were already built, we use the following notations: $\left\|a_{i_j^\alpha j_i^\beta}\right\|_{i\in I}^{j\in J} \equiv \left\|a_{ij}^{\alpha\beta}\right\|_{i\in I}^{j\in J}$ and $\left\|b_{i_j^\alpha j_i^\beta}\right\|_{i\in I}^{j\in J} \equiv \left\|b_{ij}^{\alpha\beta}\right\|_{i\in I}^{j\in J}$ for all $\alpha = \overline{1,\chi_1}$ and $\beta = \overline{1,\chi_2}$, so we have a bimatrix game where the utility is determined by a set of matrices:

$$AB(\alpha,\beta) = \begin{pmatrix} \left(a_{11}^{\alpha\beta}, b_{11}^{\alpha\beta}\right) & \cdots & \left(a_{1j}^{\alpha\beta}, b_{1j}^{\alpha\beta}\right) & \cdots & \left(a_{1m}^{\alpha\beta}, b_{1m}^{\alpha\beta}\right) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \left(a_{i1}^{\alpha\beta}, b_{i1}^{\alpha\beta}\right) & \cdots & \left(a_{ij}^{\alpha\beta}, b_{ij}^{\alpha\beta}\right) & \cdots & \left(a_{im}^{\alpha\beta}, b_{im}^{\alpha\beta}\right) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \left(a_{nm}^{\alpha\beta}, b_{nm}^{\alpha\beta}\right) & \cdots & \left(a_{nj}^{\alpha\beta}, b_{nj}^{\alpha\beta}\right) & \cdots & \left(a_{nm}^{\alpha\beta}, b_{nm}^{\alpha\beta}\right) \end{pmatrix}$$

for $\alpha = \overline{1,\chi_1}$ and $\beta = \overline{1,\chi_2}$ and the set of strategies are I and J. Every player knows that the utilities are determined by the set of matrices $\left\{AB(\alpha,\beta) = \left\|\left(a_{ij}^{\alpha\beta}, b_{ij}^{\alpha\beta}\right)\right\|_{i\in I}^{j\in J}\right\}_{\alpha=\overline{1,\chi_1}}^{\beta=\overline{1,\chi_2}}$, but they do not know which matrix from this set will be used.

So, the bimatrix game $Game(1 \leftrightarrow 2)$ of imperfect information on the set of informational extended strategies generates the following normal form incomplete information game on the sets of non-informational extended strategies I, J

$$\tilde{\mathbf{\Gamma}} = \left\langle \{1,2\}, I, J, \left\{ AB(\alpha,\beta) = \left\| \left(a_{ij}^{\alpha\beta}, b_{ij}^{\alpha\beta} \right) \right\|_{i\in I}^{j\in J} \right\}_{\alpha=\overline{1,\chi_1}}^{\beta=\overline{1,\chi_2}} \right\rangle. \qquad (6.4.5)$$

In the game theory, it is standard to begin analyses with the assumption that players are Bayesian rational. The way to modelling this situation of **asymmetric** or **incomplete** informations by recurring to an idea generated by Harsanyi(1967). The key is to introduce a move by the Nature, which transforms the uncertainty by converting an **incomplete information** problem into an **imperfect information problem**. The idea is that the Nature moves determining player's types, a concept that embodies all the relevant private information about them (such as payoffs, preferences, beliefs about other players, etc.). Harsanyi described a game as having incomplete information when the players are uncertain about each other's types.

According to [Hancu 2014] we can construct the bimatrix Bayesian game for the bimatrix incomplete information game $\tilde{\mathbf{\Gamma}}$ from (6.4.5) that consists of the following.

1. A set of players {1,2};
2. A set of possible actions for each player: for player 1 is $I = \{1,2,..,n\}$, the line index, and for player 2 is $J = \{1,2,..,m\}$, the column index;
3. A set of possible types for each player that coincides with the set of informational extended strategies of that player, namely $\Theta_1 = \{\theta_1^\alpha : \mathrm{J} \to \mathrm{I}\}_{\alpha=1}^{\chi_1}$ for player 1 and respectively $\Theta_2 = \left\{\theta_2^\beta : \mathrm{I} \to \mathrm{J}\right\}_{\beta=1}^{\chi_2}$ for the player 2. So the types of the player 1 are $\Delta_1 = \{\alpha = 1, \ldots, \chi_1\}$ and of the player 2 are $\Delta_2 = \{\beta = 1, \ldots, \chi_2\}$. Only player 1(player 2) knows his type α (type β) when play begins.
4. A probability function that specifies, for each possible type of each player, a probability distribution over the other player's possible types, describing what each type of each player would believe about the other players' types $\mathrm{p}: \Delta_1 \to \Omega(\Delta_2)$, $\mathrm{q}: \Delta_2 \to \Omega(\Delta_1)$,where $\Omega(\Delta_2)$ (respectively $\Omega(\Delta_1)$) denotes the set of all probability distributions on a set Δ_1 (respectively Δ_2). The function p (respectively q) summarizes what player 1 (respectively player 2), given his type, believes about the types of the other players. So, $\mathrm{p}(\beta|\alpha) = \frac{\mathrm{p}(\beta\cap\alpha)}{\mathrm{p}(\alpha)}$ (Bayes'Rule) (respectively $\mathrm{q}(\alpha|\beta) = \frac{\mathrm{q}(\alpha\cap\beta)}{\mathrm{q}(\beta)}$) is the conditional probability assigned to the type $\beta \in \Delta_2$ (respectively $\alpha \in \Delta_1$) when the type of the player 1 is α (respectively of the player 2 is β).

5. Combining actions and types for each player it is possible to construct the strategies. Strategies will be given by a mapping from the type space to the action space. In other words, a strategy may assign different actions to different types. The sets of pure strategies of the players (line and columns) will depend on the type of the players (or, in other words, on what informational extended strategy will chose the players). So, in this way, we will construct the strategies of the players. If player 1 is of type $\alpha \in \Delta_1$ and player 1 knows that the type of the player 2 may be an element from the set $\Delta_2 = \{\beta = 1, \dots, \chi_2\}$, and because the utility matrix elements also depend on the type β of player 2, then the set of matrices that represent his utility is $\left\{A(\alpha,\beta) = \left\|a_{ij}^{\alpha\beta}\right\|_{i\in I}^{j\in J}\right\}_{\beta=\overline{1,\chi_2}}$. We will denote the pure strategy of player 1 by $\tilde{\imath} = i_1 i_2 \dots i_\beta \dots i_{\kappa_2}$ and it has the following meaning: the player will chose the line $i_1 \in I$ if $\beta = 1$, namely line i_1 from the utility matrix $A(\alpha, 1)$ and line $i_2 \in I$ if $\beta = 2$ and so on, line $i_{\chi_2} \in I$ if $\beta = \chi_2$. Then the set of all pure strategy of player 1 will be determined by the set of all corteges of type $i_1 i_2 \dots i_\beta \dots i_{\chi_2}$ for all $i_\beta \in I$ and will be denoted by $\tilde{I}(\alpha)$. In his turn, if player 2 is of type $\beta \in \Delta_2$ and he knows that the type of player 1 may be an element from the set $\Delta_1 = \{\alpha = 1, \dots, \chi_1\}$, and because the utility matrix elements depend also on the type α of player 1, then the set of matrices that represent his utility is $\left\{B(\alpha,\beta) = \left\|b_{ij}^{\alpha\beta}\right\|_{i\in I}^{j\in J}\right\}_{\alpha=\overline{1,\chi_1}}$. By the same way we will denote the pure strategy of player 2 by $\tilde{\jmath} = j_1 j_2 \dots j_\alpha \dots j_{\chi_2}$ and it has the following meaning: the player will chose column $j_1 \in J$ if $\alpha = 1$, namely column j_1 from utility matrix $B(1,\beta)$ and column $j_2 \in J$ if $\alpha = 2$ and so on he will chose column $j_{\chi_1} \in J$ if $\alpha = \chi_1$. Then the set of all pure strategy of player 2 will be determined by the set of all corteges of type $j_1 j_2 \dots j_\alpha \dots j_{\chi_2}$ for all $j_\alpha \in J$ and will be denoted by $\tilde{J}(\beta)$.

6. A payoff function specifies each player's expected payoff matrices for every possible combination of all player's actions and types. Hence, if the player 1 of type α chooses the pure strategy $\tilde{\imath} \in \tilde{I}(\alpha)$, and the player 2 plays some strategy $\tilde{\jmath} \in \tilde{J}(\hat{a})$ for all $\beta \in \Delta_2$, then expected payoffs of player 1 is the following matrix

$$\mathbf{A}(\alpha) = \left\|\mathbf{a}_{\tilde{\imath}\tilde{\jmath}}\right\|_{\tilde{\imath}\in\tilde{\mathbf{I}}(\alpha)}^{\tilde{\jmath}\in\tilde{\mathbf{J}}(\beta)} \qquad (6.4.6)$$

where $\mathbf{a}_{\tilde{\imath}\tilde{\jmath}} = \sum_{\beta\in\Delta_2} p(\beta/\alpha) a_{i_\beta j_\alpha}^{\alpha\beta}$. Similarly, if player 2 of type β chooses the pure strategy $\tilde{\mathbf{j}} \in \tilde{\mathbf{J}}(\beta)$ and the player 1 plays some strategy $\tilde{\mathbf{i}} \in \tilde{\mathbf{I}}(\alpha)$ for all $\alpha \in \Delta_1$, then expected payoffs of player 2 of type β is

$$\mathbf{B}(\beta) = \left\| \mathbf{b}_{\tilde{\mathbf{i}}\tilde{\mathbf{j}}} \right\|_{\tilde{\mathbf{i}} \in \tilde{\mathbf{I}}(\alpha)}^{\tilde{\mathbf{j}} \in \tilde{\mathbf{J}}(\beta)} \quad (6.4.7)$$

where $\mathbf{b}_{\tilde{\mathbf{i}}\tilde{\mathbf{j}}} = \sum_{\alpha \in \Delta_1} q(\alpha|\beta) b_{i_\beta j_\alpha}^{\alpha\beta}$.

So we can introduce the following definition.

Definition 6.4.1 *For the incomplete information game* $\widetilde{\boldsymbol{\Gamma}}$ *from (6.4.5) the normal form game*

$$\Gamma_{Bayes} = \langle \{1,2\}, \tilde{\mathbf{I}}, \tilde{\mathbf{J}}, \mathcal{A}, \mathcal{B} \rangle, \quad (6.4.8)$$

where $\tilde{\boldsymbol{I}} = \bigcup_{\alpha \in \Delta_1} \tilde{\boldsymbol{I}}(\alpha)$, $\tilde{\boldsymbol{J}} = \bigcup_{\beta \in \Delta_2} J(\alpha)$ *and the utility matrices are* $\mathcal{A} = \|\boldsymbol{A}(\alpha)\|_{\alpha \in \Delta_1}$ *and* $\mathcal{B} = \|\boldsymbol{B}(\beta)\|_{\beta \in \Delta_2}$, *is called the associated Bayesian game in the non informational extended strategies.*

It is important to discuss a little bit each part of the definition above. Players types contain all relevant information about certain player's private characteristics of the informational extended strategy to choose. The type α (respectively β) is only observed by player 1(player 2), who uses this information both to make decisions and to update his beliefs about the likelihood of opponents types (using the conditional probability $p(\beta|\alpha)$ (respectively $q(\alpha|\beta)$). We still assume common knowledge of the 1)-6) items, but we allow uncertainty about players' preferences. Player's (α, β) type determines (α, β) payoffs matrices $(\mathbf{A}(\alpha), \mathbf{B}(\beta))$.

The games defined above are sometimes called *Bayesian normal form games*, since the drawing of types is followed by a simultaneous move game. One can also define *Bayesian extensive form games*, where the drawing of types is followed by an extensive form game.

Definition 6.4.2 *(Bayesian Nash Equilibrium) The strategy profiles* $(\boldsymbol{i}^*, \boldsymbol{j}^*)$, $\boldsymbol{i}^* \in \tilde{\boldsymbol{I}}$, $\boldsymbol{j}^* \in \tilde{\boldsymbol{J}}$ *is Bayes-Nash equilibrium if we have*

$$\begin{cases} \mathbf{a}_{\mathbf{i}^*\mathbf{j}^*} \geq \mathbf{a}_{\mathbf{i}\mathbf{j}^*} & for\ all\ \mathbf{i} \in \tilde{\mathbf{I}}, \\ \mathbf{b}_{\mathbf{i}^*\mathbf{j}^*} \geq \mathbf{b}_{\mathbf{i}^*\mathbf{j}} & for\ all\ \mathbf{j} \in \tilde{\mathbf{J}}. \end{cases}$$

Denote by $BE[\Gamma_{Bayes}]$ the set of all Bayes-Nash profile of the game Γ_{Bayes} from (6.4.8).

Remark 6.4.1 *The Bayesian Game* Γ_{Bayes} *(6.4.8) for all* $\alpha \in \Delta_1$ *and* $\beta \in \Delta_2$ *is a bimatrix game where player* 1 *is of type* α *and player* 2 *is of type* β. *The Bayese-Nash equilibria profile following the Definition 6.4.2 will be found in the next way: we find the Nash equilibria profile for a bimatrix game where the sets of strategies are the "extended sets"* $\tilde{\boldsymbol{I}} = \bigcup_{\alpha \in \Delta_1} \tilde{\boldsymbol{I}}(\alpha)$, $\tilde{\boldsymbol{J}} = \bigcup_{\beta \in \Delta_2} J(\alpha)$ *and the utility matrices are the "extended matrices"* $\mathcal{A} = \|\boldsymbol{A}(\alpha)\|_{\alpha \in \Delta_1}$ *and* $\mathcal{B} = \|\boldsymbol{B}(\beta)\|_{\beta \in \Delta_2}$.

We will introduce the next definition.

Definition 6.4.3 *For all fixed* $\alpha \in \Delta_1$ *and* $\beta \in \Delta_2$ *the game* $sub\Gamma_{Bayes} =$

$\langle\{1,2\}, \tilde{\boldsymbol{I}}(\alpha), \tilde{\boldsymbol{J}}(\beta), \boldsymbol{A}(\alpha), \boldsymbol{B}(\beta)\rangle$ *will be called a subgame of the Bayesian game* Γ_{Bayes} *from (6.4.8).*

According to [Harsanyi, 1998], using the notion of "type-players", the $sub\Gamma_{Bayes}$ is the bimatrix game of the type-player α and of the type-player β.

6.4.1 Bayesian game for the 2 × 3 bimatrix games in incomplete information, generated by the informational extended strategies.

Consider a bimatrix game in incomplete information for which the utilities are:

$$AB(\alpha,\beta) = \begin{pmatrix} \left(a_{11}^{\alpha\beta}, b_{11}^{\alpha\beta}\right) & \left(a_{12}^{\alpha\beta}, b_{12}^{\alpha\beta}\right) & \left(a_{13}^{\alpha\beta}, b_{13}^{\alpha\beta}\right) \\ \left(a_{21}^{\alpha\beta}, b_{21}^{\alpha\beta}\right) & \left(a_{22}^{\alpha\beta}, b_{22}^{\alpha\beta}\right) & \left(a_{23}^{\alpha\beta}, b_{23}^{\alpha\beta}\right) \end{pmatrix}.$$

The Bayesian game will contain the elements.

- The set of players $\{1,2\}$.
- The set of actions of the players $I = \{1,2\}, J = \{1,2,3\}$.
- The set of types of the player 1 is $\Delta_1 = \{\alpha = \overline{1,8}\}$ and of the player 2 is $\Delta_2 = \{\beta = \overline{1,9}\}$.
- denote the type probability for player 1 by $p(\beta|\alpha)$, respectively $q(\alpha|\beta)$ for player 2.
- For any fixed α we introduce the notation $i_\beta i_\gamma$, for $\beta, \gamma \in \Delta_2, \beta \neq \gamma$, which satisfies the conditions: the player 1 will chose the line $i_\beta \in I$ in case if the player 2 is of type β, namely, the utility of the player is the matrix $\left\|a_{ij}^{\alpha\beta}\right\|_{i\in I}^{j\in J}$, and will chose the line $i_\gamma \in I$ if the player 2 is of type γ (namely, the utility of the player is the matrix $\left\|a_{ij}^{\alpha\gamma}\right\|_{i\in I}^{j\in J}$). Thus the set of pure strategies of the player 1 is $\tilde{\mathbf{I}}(\alpha) = \{\tilde{\mathbf{i}} = i_\beta i_\gamma : i_\beta \in I, i_\gamma \in I, \forall\beta,\gamma \in \Delta_2, \beta \neq \gamma\} = \{1_1 1_2, 1_1 2_2, 2_1 1_2, 2_1 2_2\}$. In the same way we will construct the strategies of the player 2. For any fixed β we will denote $j_\alpha j_\delta$ for $\alpha, \delta \in \Delta_1, \alpha \neq \delta$, which meaning is: the player 2 will chose the column $j_\alpha \in J$ if the player 1 is of type α, i.e. the utility of the player is the matrix $\left\|b_{ij}^{\alpha\beta}\right\|_{i\in I}^{j\in J}$, and will chose the column $j_\delta \in J$ if the player 1 is of type δ, i.e. the utility of the player is the matrix $\left\|b_{ij}^{\delta\beta}\right\|_{i\in I}^{j\in J}$. Thus the set of pure strategies of the player 2 is $\tilde{\mathbf{J}}(\beta) = \{\tilde{\mathbf{j}} = j_\alpha j_\delta : j_\alpha \in J, j_\delta \in J, \forall\alpha,\delta \in \Delta_1, \alpha \neq \delta\} = \{1_1 1_2, 1_1 2_2, 2_1 1_2, 2_1 2_2, 1_1 3_2, 3_1 1_2, 2_1 3_2, 3_1 2_2, 3_1 3_2\}$.
- The players do not know the exact type of the partners and supply this lack of information by the belief probabilities. Let $\Delta_1 = \{\alpha = 1,2\}$ and $\Delta_2 = \{\beta = 1,2\}$. Thus the player 1, being of type α, will assume with the probability $p(\beta = 1|\alpha)$ that he has the payoff matrice $\begin{pmatrix} a_{11}^{\alpha 1} & a_{12}^{\alpha 1} & a_{13}^{\alpha 1} \\ a_{21}^{\alpha 1} & a_{22}^{\alpha 1} & a_{23}^{\alpha 1} \end{pmatrix}$ and, with

probabilities $p(\beta = 2|\alpha)$, the payoff matrice$\begin{pmatrix} a_{11}^{\alpha 2} & a_{12}^{\alpha 2} & a_{13}^{\alpha 2} \\ a_{21}^{\alpha 2} & a_{22}^{\alpha 2} & a_{23}^{\alpha 2} \end{pmatrix}$. Respectively, the player 2, being of type β, will assume with the probability $q(\alpha = 1|\beta)$ that he has the payoff matrice$\begin{pmatrix} b_{11}^{1\beta} & b_{12}^{1\beta} & b_{13}^{1\beta} \\ b_{21}^{1\beta} & b_{22}^{1\beta} & b_{23}^{1\beta} \end{pmatrix}$ and, with the probability $q(\alpha = 2|\beta)$, the payoff matrice$\begin{pmatrix} b_{11}^{2\beta} & b_{11}^{2\beta} & b_{13}^{2\beta} \\ b_{21}^{2\beta} & b_{22}^{2\beta} & b_{23}^{2\beta} \end{pmatrix}$.

We denote by

$$E_1(a_{i_1 j_\alpha}^{\alpha 1}, a_{i_2 j_\alpha}^{\alpha 2}) = p(\beta = 1|\alpha) a_{i_1 j_\alpha}^{\alpha 1} + p(\beta = 2|\alpha) a_{i_2 j_\alpha}^{\alpha 2}$$

$$E_2(b_{i_\beta j_1}^{1\beta}, b_{i_\beta j_2}^{2\beta}) = q(\alpha = 1|\beta) b_{i_\beta j_1}^{1\beta} + q(\alpha = 2|\beta) b_{i_\beta j_2}^{2\beta}$$

for any $i \in I$, $j \in J$, $\alpha \in \Delta_1$, $\beta \in \Delta_2$, the average value if the player 1, respectively the player 2, knows the belief probabilities (or the probabilities setted by the Nature). We will construct the utility matrices when the player 1 is of type α and, at the same time, the player 2 is of type β. Based on the facts mentioned above we will obtain the next bimatrix game in which the utility of the players is described by the following matrices with 4 lines and 9 columns:

A(α)=

ĩ\j̃	$1_1 1_2$	$1_1 2_2$	$2_1 1_2$	$2_1 2_2$	$1_1 3_2$
$1_1 1_2$	$E_1(a_{1j_\alpha}^{\alpha 1}, a_{1j_\alpha}^{\alpha 2})$	$E_1(a_{1j_\alpha}^{\alpha 1}, a_{1j_\alpha}^{\alpha 2})$	$E_1(a_{1j_\alpha}^{\alpha 1}, a_{1j_\alpha}^{\alpha 2})$	$E_1(a_{1j_\alpha}^{\alpha 1}, a_{1j_\alpha}^{\alpha 2})$	$E_1(a_{1j_\alpha}^{\alpha 1}, a_{1j_\alpha}^{\alpha 2})$
$1_1 2_2$	$E_1(a_{1j_\alpha}^{\alpha 1}, a_{2j_\alpha}^{\alpha 2})$	$E_1(a_{1j_\alpha}^{\alpha 1}, a_{2j_\alpha}^{\alpha 2})$	$E_1(a_{1j_\alpha}^{\alpha 1}, a_{2j_\alpha}^{\alpha 2})$	$E_1(a_{1j_\alpha}^{\alpha 1}, a_{2j_\alpha}^{\alpha 2})$	$E_1(a_{1j_\alpha}^{\alpha 1}, a_{2j_\alpha}^{\alpha 2})$
$2_1 1_2$	$E_1(a_{2j_\alpha}^{\alpha 1}, a_{1j_\alpha}^{\alpha 2})$	$E_1(a_{2j_\alpha}^{\alpha 1}, a_{1j_\alpha}^{\alpha 2})$	$E_1(a_{2j_\alpha}^{\alpha 1}, a_{1j_\alpha}^{\alpha 2})$	$E_1(a_{2j_\alpha}^{\alpha 1}, a_{1j_\alpha}^{\alpha 2})$	$E_1(a_{2j_\alpha}^{\alpha 1}, a_{1j_\alpha}^{\alpha 2})$
$2_1 2_2$	$E_1(a_{2j_\alpha}^{\alpha 1}, a_{2j_\alpha}^{\alpha 2})$	$E_1(a_{2j_\alpha}^{\alpha 1}, a_{2j_\alpha}^{\alpha 2})$	$E_1(a_{2j_\alpha}^{\alpha 1}, a_{2j_\alpha}^{\alpha 2})$	$E_1(a_{2j_\alpha}^{\alpha 1}, a_{2j_\alpha}^{\alpha 2})$	$E_1(a_{2j_\alpha}^{\alpha 1}, a_{2j_\alpha}^{\alpha 2})$

continued

ĩ\j̃	$3_1 1_2$	$2_1 3_2$	$3_1 2_2$	$3_1 3_2$
$1_1 1_2$	$E_1(a_{1j_\alpha}^{\alpha 1}, a_{1j_\alpha}^{\alpha 2})$	$E_1(a_{1j_\alpha}^{\alpha 1}, a_{1j_\alpha}^{\alpha 2})$	$E_1(a_{1j_\alpha}^{\alpha 1}, a_{1j_\alpha}^{\alpha 2})$	$E_1(a_{1j_\alpha}^{\alpha 1}, a_{1j_\alpha}^{\alpha 2})$
$1_1 2_2$	$E_1(a_{1j_\alpha}^{\alpha 1}, a_{2j_\alpha}^{\alpha 2})$	$E_1(a_{1j_\alpha}^{\alpha 1}, a_{2j_\alpha}^{\alpha 2})$	$E_1(a_{1j_\alpha}^{\alpha 1}, a_{2j_\alpha}^{\alpha 2})$	$E_1(a_{1j_\alpha}^{\alpha 1}, a_{2j_\alpha}^{\alpha 2})$
$2_1 1_2$	$E_1(a_{2j_\alpha}^{\alpha 1}, a_{1j_\alpha}^{\alpha 2})$	$E_1(a_{2j_\alpha}^{\alpha 1}, a_{1j_\alpha}^{\alpha 2})$	$E_1(a_{2j_\alpha}^{\alpha 1}, a_{1j_\alpha}^{\alpha 2})$	$E_1(a_{2j_\alpha}^{\alpha 1}, a_{1j_\alpha}^{\alpha 2})$
$2_1 2_2$	$E_1(a_{2j_\alpha}^{\alpha 1}, a_{2j_\alpha}^{\alpha 2})$	$E_1(a_{2j_\alpha}^{\alpha 1}, a_{2j_\alpha}^{\alpha 2})$	$E_1(a_{2j_\alpha}^{\alpha 1}, a_{2j_\alpha}^{\alpha 2})$	$E_1(a_{2j_\alpha}^{\alpha 1}, a_{2j_\alpha}^{\alpha 2})$

ĩ\j̃	$1_1 1_2$	$1_1 2_2$	$2_1 1_2$	$2_1 2_2$	$1_1 3_2$
$1_1 1_2$	$E_2\left(b_{i_\beta 1}^{1\beta}, b_{i_\beta 1}^{2\beta}\right)$	$E_2\left(b_{i_\beta 1}^{1\beta}, b_{i_\beta 2}^{2\beta}\right)$	$E_2\left(b_{i_\beta 2}^{1\beta}, b_{i_\beta 1}^{2\beta}\right)$	$E_2\left(b_{i_\beta 2}^{1\beta}, b_{i_\beta 2}^{2\beta}\right)$	$E_2\left(b_{i_\beta 1}^{1\beta}, b_{i_\beta 3}^{2\beta}\right)$
$1_1 2_2$	$E_2\left(b_{i_\beta 1}^{1\beta}, b_{i_\beta 1}^{2\beta}\right)$	$E_2\left(b_{i_\beta 1}^{1\beta}, b_{i_\beta 2}^{2\beta}\right)$	$E_2\left(b_{i_\beta 2}^{1\beta}, b_{i_\beta 1}^{2\beta}\right)$	$E_2\left(b_{i_\beta 2}^{1\beta}, b_{i_\beta 2}^{2\beta}\right)$	$E_2\left(b_{i_\beta 1}^{1\beta}, b_{i_\beta 3}^{2\beta}\right)$

$2_1 1_2$	$E_2\left(b^{1\beta}_{i_\beta 1}, b^{2\beta}_{i_\beta 1}\right)$	$E_2\left(b^{1\beta}_{i_\beta 1}, b^{2\beta}_{i_\beta 2}\right)$	$E_2\left(b^{1\beta}_{i_\beta 2}, b^{2\beta}_{i_\beta 1}\right)$	$E_2\left(b^{1\beta}_{i_\beta 2}, b^{2\beta}_{i_\beta 2}\right)$	$E_2\left(b^{1\beta}_{i_\beta 1}, b^{2\beta}_{i_\beta 3}\right)$
$2_1 2_2$	$E_2\left(b^{1\beta}_{i_\beta 1}, b^{2\beta}_{i_\beta 1}\right)$	$E_2\left(b^{1\beta}_{i_\beta 1}, b^{2\beta}_{i_\beta 2}\right)$	$E_2\left(b^{1\beta}_{i_\beta 2}, b^{2\beta}_{i_\beta 1}\right)$	$E_2\left(b^{1\beta}_{i_\beta 2}, b^{2\beta}_{i_\beta 2}\right)$	$E_2\left(b^{1\beta}_{i_\beta 1}, b^{2\beta}_{i_\beta 3}\right)$

B(β)=

continued

ĩ\j̃	$3_1 1_2$	$2_1 3_2$	$3_1 2_2$	$3_1 3_2$
$1_1 1_2$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 1}\right)$	$E_2\left(b^{1\beta}_{i_\beta 2}, b^{2\beta}_{i_\beta 3}\right)$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 2}\right)$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 3}\right)$
$1_1 2_2$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 1}\right)$	$E_2\left(b^{1\beta}_{i_\beta 2}, b^{2\beta}_{i_\beta 3}\right)$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 2}\right)$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 3}\right)$
$2_1 1_2$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 1}\right)$	$E_2\left(b^{1\beta}_{i_\beta 2}, b^{2\beta}_{i_\beta 3}\right)$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 2}\right)$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 3}\right)$
$2_1 2_2$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 1}\right)$	$E_2\left(b^{1\beta}_{i_\beta 2}, b^{2\beta}_{i_\beta 3}\right)$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 2}\right)$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 3}\right)$

What is the meaning, for example, of elements at the intersection of line $1_1 2_2$ and column $1_1 3_2$? Using the belief probabilities for types of the players, we get that player 1, being of type α, will chose the line $i = 1$ from the matrix $\begin{pmatrix} a^{\alpha 1}_{11} & a^{\alpha 1}_{12} & a^{\alpha 1}_{13} \\ a^{\alpha 1}_{21} & a^{\alpha 1}_{22} & a^{\alpha 1}_{23} \end{pmatrix}$ (when the player 2 is of type $\beta = 1$) and line $i = 2$ from the matrix $\begin{pmatrix} a^{\alpha 2}_{11} & a^{\alpha 2}_{12} & a^{\alpha 2}_{13} \\ a^{\alpha 2}_{21} & a^{\alpha 2}_{22} & a^{\alpha 2}_{23} \end{pmatrix}$ (when player 2 is of type $\beta = 2$), and correspondingly, the player 2 being of type β, will chose the column $j = 1$ from the matrix $\begin{pmatrix} b^{1\beta}_{11} & b^{1\beta}_{12} & b^{1\beta}_{13} \\ b^{1\beta}_{21} & b^{1\beta}_{22} & b^{1\beta}_{23} \end{pmatrix}$ (when the type of the player 1 is $\alpha = 1$) and the column $j = 3$ in the matrix $\begin{pmatrix} b^{2\beta}_{11} & b^{2\beta}_{12} & b^{2\beta}_{13} \\ b^{2\beta}_{21} & b^{2\beta}_{22} & b^{2\beta}_{23} \end{pmatrix}$ (when the type of the player 1 is $\alpha = 2$), then the average value of the payoff for the player 1 is

$$E_1\left(a^{\alpha 1}_{1 j_\alpha}, a^{\alpha 2}_{2 j_\alpha}\right) = p(\beta = 1|\alpha) a^{\alpha 1}_{1 j_\alpha} + p(\beta = 2|\alpha) a^{\alpha 2}_{2 j_\alpha}$$

and respectively, for the player 2 is

$$E_2\left(b^{1\beta}_{i_\beta 1}, b^{2\beta}_{i_\beta 3}\right) = q(\alpha = 1|\beta) b^{1\beta}_{i_\beta 1} + q(\alpha = 2|\beta) b^{2\beta}_{i_\beta 3}.$$

Finally we will obtain for $\alpha = 1, \alpha = 2, \beta = 1$ and $\beta = 2$

ĩ\j̃	$1_1 1_2$	$1_1 2_2$	$2_1 1_2$	$2_1 2_2$	$1_1 3_2$
$1_1 1_2$	$E_1(a^{11}_{11}, a^{12}_{11})$	$E_1(a^{11}_{11}, a^{12}_{11})$	$E_1(a^{11}_{12}, a^{12}_{12})$	$E_1(a^{11}_{12}, a^{12}_{12})$	$E_1(a^{11}_{11}, a^{12}_{11})$
$1_1 2_2$	$E_1(a^{11}_{11}, a^{12}_{21})$	$E_1(a^{11}_{11}, a^{12}_{21})$	$E_1(a^{11}_{12}, a^{12}_{22})$	$E_1(a^{11}_{12}, a^{12}_{22})$	$E_1(a^{11}_{11}, a^{12}_{21})$
$2_1 1_2$	$E_1(a^{11}_{21}, a^{12}_{11})$	$E_1(a^{11}_{21}, a^{12}_{11})$	$E_1(a^{11}_{22}, a^{12}_{12})$	$E_1(a^{11}_{22}, a^{12}_{12})$	$E_1(a^{11}_{21}, a^{12}_{11})$
$2_1 2_2$	$E_1(a^{11}_{21}, a^{12}_{21})$	$E_1(a^{11}_{21}, a^{12}_{21})$	$E_1(a^{11}_{22}, a^{12}_{22})$	$E_1(a^{11}_{22}, a^{12}_{22})$	$E_1(a^{11}_{21}, a^{12}_{21})$

A(1)=

continued

ĩ\j̃	$3_1 1_2$	$2_1 3_2$	$3_1 2_2$	$3_1 3_2$
$1_1 1_2$	$E_1(a^{11}_{13}, a^{12}_{13})$	$E_1(a^{11}_{12}, a^{12}_{12})$	$E_1(a^{11}_{13}, a^{12}_{13})$	$E_1(a^{11}_{13}, a^{12}_{13})$
$1_1 2_2$	$E_1(a^{11}_{13}, a^{12}_{23})$	$E_1(a^{11}_{12}, a^{12}_{22})$	$E_1(a^{11}_{13}, a^{12}_{23})$	$E_1(a^{11}_{23}, a^{12}_{13})$
$2_1 1_2$	$E_1(a^{11}_{11}, a^{12}_{11})$	$E_1(a^{11}_{11}, a^{12}_{11})$	$E_1(a^{11}_{11}, a^{12}_{11})$	$E_1(a^{11}_{11}, a^{12}_{11})$
$2_1 2_2$	$E_1(a^{11}_{23}, a^{12}_{23})$	$E_1(a^{11}_{22}, a^{12}_{22})$	$E_1(a^{11}_{23}, a^{12}_{23})$	$E_1(a^{11}_{23}, a^{12}_{23})$

A(2)=

$\tilde{\mathbf{i}}\backslash\tilde{\mathbf{j}}$	$1_1 1_2$	$1_1 2_2$	$2_1 1_2$	$2_1 2_2$	$1_1 3_2$
$1_1 1_2$	$E_1(a_{11}^{21}, a_{11}^{22})$	$E_1(a_{12}^{21}, a_{12}^{22})$	$E_1(a_{11}^{21}, a_{11}^{22})$	$E_1(a_{12}^{21}, a_{12}^{22})$	$E_1(a_{13}^{21}, a_{13}^{22})$
$1_1 2_2$	$E_1(a_{11}^{21}, a_{21}^{22})$	$E_1(a_{12}^{21}, a_{22}^{22})$	$E_1(a_{11}^{21}, a_{21}^{22})$	$E_1(a_{12}^{21}, a_{22}^{22})$	$E_1(a_{13}^{21}, a_{23}^{22})$
$2_1 1_2$	$E_1(a_{21}^{21}, a_{11}^{22})$	$E_1(a_{22}^{21}, a_{22}^{22})$	$E_1(a_{21}^{21}, a_{11}^{22})$	$E_1(a_{22}^{21}, a_{12}^{22})$	$E_1(a_{23}^{21}, a_{13}^{22})$
$2_1 2_2$	$E_1(a_{21}^{21}, a_{21}^{22})$	$E_1(a_{22}^{21}, a_{22}^{22})$	$E_1(a_{21}^{21}, a_{21}^{22})$	$E_1(a_{22}^{21}, a_{22}^{22})$	$E_1(a_{23}^{21}, a_{23}^{22})$

continued

$\tilde{\mathbf{i}}\backslash\tilde{\mathbf{j}}$	$3_1 1_2$	$2_1 3_2$	$3_1 2_2$	$3_1 3_2$
$1_1 1_2$	$E_1(a_{11}^{21}, a_{11}^{22})$	$E_1(a_{13}^{21}, a_{13}^{22})$	$E_1(a_{12}^{21}, a_{12}^{22})$	$E_1(a_{13}^{21}, a_{13}^{22})$
$1_1 2_2$	$E_1(a_{11}^{21}, a_{21}^{22})$	$E_1(a_{13}^{21}, a_{23}^{22})$	$E_1(a_{12}^{21}, a_{22}^{22})$	$E_1(a_{13}^{21}, a_{23}^{22})$
$2_1 1_2$	$E_1(a_{21}^{21}, a_{11}^{22})$	$E_1(a_{23}^{21}, a_{13}^{22})$	$E_1(a_{22}^{21}, a_{12}^{22})$	$E_1(a_{23}^{21}, a_{13}^{22})$
$2_1 2_2$	$E_1(a_{21}^{21}, a_{21}^{22})$	$E_1(a_{23}^{21}, a_{23}^{22})$	$E_1(a_{22}^{21}, a_{22}^{22})$	$E_1(a_{23}^{21}, a_{23}^{22})$

B(1)=

$\tilde{\mathbf{i}}\backslash\tilde{\mathbf{j}}$	$1_1 1_2$	$1_1 2_2$	$2_1 1_2$	$2_1 2_2$	$1_1 3_2$
$1_1 1_2$	$E_2(b_{11}^{11}, b_{11}^{21})$	$E_2(b_{11}^{11}, b_{12}^{21})$	$E_2(b_{12}^{11}, b_{11}^{21})$	$E_2(b_{12}^{11}, b_{12}^{21})$	$E_2(b_{11}^{11}, b_{13}^{21})$
$1_1 2_2$	$E_2(b_{11}^{11}, b_{11}^{21})$	$E_2(b_{11}^{11}, b_{12}^{21})$	$E_2(b_{12}^{11}, b_{11}^{21})$	$E_2(b_{12}^{11}, b_{12}^{21})$	$E_2(b_{11}^{11}, b_{13}^{21})$
$2_1 1_2$	$E_2(b_{21}^{11}, b_{21}^{21})$	$E_2(b_{21}^{11}, b_{22}^{21})$	$E_2(b_{22}^{11}, b_{21}^{21})$	$E_2(b_{22}^{11}, b_{22}^{21})$	$E_2(b_{21}^{11}, b_{23}^{21})$
$2_1 2_2$	$E_2(b_{21}^{11}, b_{21}^{21})$	$E_2(b_{21}^{11}, b_{22}^{21})$	$E_2(b_{22}^{11}, b_{21}^{21})$	$E_2(b_{22}^{11}, b_{22}^{21})$	$E_2(b_{21}^{11}, b_{23}^{21})$

continued

$\tilde{\mathbf{i}}\backslash\tilde{\mathbf{j}}$	$3_1 1_2$	$2_1 3_2$	$3_1 2_2$	$3_1 3_2$
$1_1 1_2$	$E_2(b_{13}^{11}, b_{11}^{21})$	$E_2(b_{12}^{11}, b_{13}^{21})$	$E_2(b_{13}^{11}, b_{12}^{21})$	$E_2(b_{13}^{11}, b_{13}^{21})$
$1_1 2_2$	$E_2(b_{13}^{11}, b_{11}^{21})$	$E_2(b_{12}^{11}, b_{13}^{21})$	$E_2(b_{13}^{11}, b_{12}^{21})$	$E_2(b_{13}^{11}, b_{13}^{21})$
$2_1 1_2$	$E_2(b_{23}^{11}, b_{21}^{21})$	$E_2(b_{22}^{11}, b_{23}^{21})$	$E_2(b_{23}^{11}, b_{22}^{21})$	$E_2(b_{23}^{11}, b_{23}^{21})$
$2_1 2_2$	$E_2(b_{23}^{11}, b_{21}^{21})$	$E_2(b_{22}^{11}, b_{23}^{21})$	$E_2(b_{23}^{11}, b_{22}^{21})$	$E_2(b_{23}^{11}, b_{23}^{21})$

B(2)=

$\tilde{\mathbf{i}}\backslash\tilde{\mathbf{j}}$	$1_1 1_2$	$1_1 2_2$	$2_1 1_2$	$2_1 2_2$	$1_1 3_2$
$1_1 1_2$	$E_2(b_{11}^{12}, b_{11}^{22})$	$E_2(b_{11}^{12}, b_{12}^{22})$	$E_2(b_{12}^{12}, b_{11}^{22})$	$E_2(b_{12}^{12}, b_{12}^{22})$	$E_2(b_{11}^{12}, b_{13}^{22})$
$1_1 2_2$	$E_2(b_{21}^{12}, b_{21}^{22})$	$E_2(b_{21}^{12}, b_{22}^{22})$	$E_2(b_{22}^{12}, b_{21}^{22})$	$E_2(b_{22}^{12}, b_{22}^{22})$	$E_2(b_{21}^{12}, b_{23}^{22})$
$2_1 1_2$	$E_2(b_{11}^{12}, b_{11}^{22})$	$E_2(b_{11}^{12}, b_{12}^{22})$	$E_2(b_{12}^{12}, b_{11}^{22})$	$E_2(b_{12}^{12}, b_{12}^{22})$	$E_2(b_{11}^{12}, b_{13}^{22})$
$2_1 2_2$	$E_2(b_{21}^{12}, b_{21}^{22})$	$E_2(b_{21}^{12}, b_{22}^{22})$	$E_2(b_{22}^{12}, b_{21}^{22})$	$E_2(b_{22}^{12}, b_{22}^{22})$	$E_2(b_{21}^{12}, b_{23}^{22})$

continued

$\tilde{\mathbf{i}}\backslash\tilde{\mathbf{j}}$	$3_1 1_2$	$2_1 3_2$	$3_1 2_2$	$3_1 3_2$
$1_1 1_2$	$E_2(b_{13}^{12}, b_{11}^{22})$	$E_2(b_{12}^{12}, b_{13}^{22})$	$E_2(b_{13}^{12}, b_{12}^{22})$	$E_2(b_{13}^{12}, b_{13}^{22})$
$1_1 2_2$	$E_2(b_{23}^{12}, b_{21}^{22})$	$E_2(b_{22}^{12}, b_{23}^{22})$	$E_2(b_{23}^{12}, b_{22}^{22})$	$E_2(b_{23}^{12}, b_{23}^{22})$
$2_1 1_2$	$E_2(b_{13}^{12}, b_{11}^{22})$	$E_2(b_{12}^{12}, b_{13}^{22})$	$E_2(b_{13}^{12}, b_{12}^{22})$	$E_2(b_{13}^{12}, b_{13}^{22})$
$2_1 2_2$	$E_2(b_{23}^{12}, b_{21}^{22})$	$E_2(b_{22}^{12}, b_{23}^{22})$	$E_2(b_{23}^{12}, b_{22}^{22})$	$E_2(b_{23}^{12}, b_{23}^{22})$

So,the normal form of the Bayesian game is

$$\Gamma_{Bayes} = \langle \{1,2\}, \tilde{\mathbf{I}} = \tilde{\mathbf{I}}(\alpha = 1) \cup \tilde{\mathbf{I}}(\alpha = 2), \tilde{\mathbf{J}} = \tilde{\mathbf{J}}(\beta = 1) \cup \tilde{\mathbf{J}}(\beta = 1),$$

$$\mathcal{A} = \|\mathbf{A}(\alpha = 1), \mathbf{A}(\alpha = 1)\|, \mathcal{B} = \|\mathbf{B}(\beta = 1), \mathbf{B}(\beta = 2)\| \quad \rangle.$$

Bimatrix games $\langle A(1), B(1) \rangle$, $\langle A(1), B(2) \rangle$, $\langle A(2), B(1) \rangle$ and $\langle A(2), B(2) \rangle$ are subgames of the constructed above Bayesian game.

As a particular case we will examine the next example. We consider the following bimatrix game $H_1 = \begin{pmatrix} 3 & 5 & 4 \\ 6 & 7 & 2 \end{pmatrix}$, $H_2 = \begin{pmatrix} 0 & 5 & 1 \\ 4 & 3 & 2 \end{pmatrix}$ for which we construct the normal form of the Bayesian game associated to the informational extended game.

For example, suppose that the informational extended strategies of the player 1 are $\theta_1^1(j) = \begin{cases} 1 \text{ if } j = 1,2 \\ 2 \text{ if } j = 3 \end{cases}$, $\theta_1^2(j) = \begin{cases} 1 \text{ if } j = 1,3 \\ 2 \text{ if } j = 2 \end{cases}$ and respectively, for the player 2 are $\theta_2^1(i) = \begin{cases} 1 \text{ if } i = 1 \\ 2 \text{ if } i = 2 \end{cases}$, $\theta_2^2(i) = \begin{cases} 1 \text{ if } i = 2 \\ 2 \text{ if } i = 1 \end{cases}$.

As mentioned above, the informational extended strategies $\{\theta_1^1, \theta_1^2, \theta_2^1, \theta_2^2\}$ generate an incomplete information game in which the payoff matrix may be one of the following matrices (one in which the utility of the players is determined by one of the matrix bellow):

$$AB(\theta_1^1, \theta_2^1) = \begin{pmatrix} (3,0) & (3,0) & (6,4) \\ (5,5) & (5,5) & (7,3) \end{pmatrix}, [see\ \Gamma(\theta_1^3, \theta_2^4)\ from\ Example\ 6.3.1]$$

$$AB(\theta_1^2, \theta_2^1) = \begin{pmatrix} (3,0) & (6,4) & (3,0) \\ (5,5) & (7,3) & (5,5) \end{pmatrix},$$

$$AB(\theta_1^1, \theta_2^2) = \begin{pmatrix} (5,5) & (5,5) & (7,3) \\ (3,0) & (3,0) & (6,0) \end{pmatrix},$$

$$AB(\theta_1^2, \theta_2^2) = \begin{pmatrix} (5,5) & (7,3) & (5,5) \\ (3,0) & (6,4) & (3,0) \end{pmatrix}.[\ see\ \Gamma(\theta_1^4, \theta_2^5)\ rom\ Example\ 6.3.1$$

We will construct the Bayesian game for the game in incomplete and imperfect information over the set of informational non extended strategies $I, J,$ which is described by the set of matrices $\{AB(\theta_1^1, \theta_2^1), AB(\theta_1^2, \theta_2^1), AB(\theta_1^1, \theta_2^2), AB(\theta_1^2, \theta_2^2)\}$. The set of types of the player 1 is $\alpha \in \Delta_1 = \{1,2\}$ and of the player 2is $\beta \in \Delta_2 = \{1,2\}$. Let's consider that the belief probabilities of the types are: for the player 1: $p(\beta|\alpha) = \begin{cases} p \text{ for } \beta = 1 \\ 1 - p \text{ for } \beta = 2 \end{cases}$ and for the player 2: $q(\alpha|\beta) = \begin{cases} q \text{ for } \alpha = 1 \\ 1 - q \text{ for } \alpha = 2 \end{cases}$, $0 \leq p \leq 1$, $0 \leq q \leq 1$. Thus we get a Bayesian game in which the utility functions of the players, depending of their types, will be:

$$\mathbf{A}(\alpha = 1) =$$

$$= \begin{pmatrix} 5-2p & 5-2p & 5-2p & 5-2p & 5-2p & 6 & 3 & 6 & 6 \\ 3 & 3 & 3 & 3 & 3 & 6p & 6-3p & 3+3p & 6 \\ 5 & 5 & 5 & 5 & 5 & 7 & 5 & 7 & 7 \\ 3+2p & 3+2p & 3+2p & 3+2p & 3+2p & 6+p & 3+2p & 6+p & 6+p \end{pmatrix},$$

$$\mathbf{B}(\beta=1)=\begin{pmatrix}0 & 4 & 0 & 4 & 0 & 4q & 0 & 4 & 4q\\ 0 & 4 & 0 & 4 & 0 & 4q & 0 & 4 & 4q\\ 5 & 3+2q & 5 & 3+2q & 5 & 5-q & 5 & 3 & 5-2q\\ 5 & 3+2q & 5 & 3+2q & 5 & 5-2q & 5 & 3 & 5-2q\end{pmatrix},$$

$$\mathbf{A}(\alpha=2)=$$

$$=\begin{pmatrix}5-2p & 7-p & 5-2p & 7-p & 5-2p & 5-2p & 5-2p & 7-p & 5-2p\\ 3 & 6 & 3 & 6 & 3 & 3 & 3 & 6 & 3\\ 3+2p & 7 & 5 & 7-4p & 5 & 5 & 5 & 7 & 5\\ 3+2p & 6+p & 3+2p & 3+2p & 6+p & 3+2p & 3+2\mathrm{p} & 6+p & 3+2p\end{pmatrix},$$

$$\mathbf{B}(\beta=2)=$$

$$=\begin{pmatrix}5 & 3+2q & 5 & 3+2q & 5 & 5-2q & 5 & 3 & 5-2q\\ 0 & 4-4q & 0 & 4-4q & 0 & 0 & 0 & 4-4q & 0\\ 5 & 3+2q & 5 & 3+2q & 5 & 5-2q & 5 & 3 & 5-2q\\ 0 & 4-4q & 0 & 4-4q & 0 & 0 & 0 & 4-4q & 0\end{pmatrix}.$$

Let $p=q=\frac{1}{2}$ then

$$\mathbf{A}(\alpha=1)=\begin{pmatrix}4 & 4 & 4 & 4 & 4 & 6 & 3 & 6 & 6\\ 3 & 3 & 3 & 3 & 3 & 3 & 9/2 & 9/2 & 6\\ 5 & 5 & 5 & 5 & 5 & 7 & 5 & 7 & 7\\ 3 & 3 & 3 & 3 & 3 & 13/2 & 3 & 13/2 & 13/2\end{pmatrix},$$

$$\mathbf{A}(\alpha=2)=\begin{pmatrix}5 & 13/2 & 5 & 13/2 & 5 & 5 & 5 & 13/2 & 5\\ 3 & 6 & 3 & 6 & 3 & 3 & 3 & 6 & 3\\ 3 & 7 & 5 & 5 & 5 & 5 & 5 & 5 & 5\\ 3 & 13/2 & 3 & 3 & 13/2 & 3 & 3 & 13/2 & 3\end{pmatrix},$$

$$\mathbf{B}(\beta=1)=\begin{pmatrix}0 & 4 & 0 & 4 & 0 & 2 & 0 & 4 & 2\\ 0 & 4 & 0 & 4 & 0 & 2 & 0 & 4 & 2\\ 5 & 3 & 5 & 3 & 5 & 9/2 & 5 & 3 & 5\\ 5 & 3 & 5 & 3 & 5 & 5 & 5 & 3 & 5\end{pmatrix},$$

$$\mathbf{B}(\beta=2)=\begin{pmatrix}5 & 3 & 5 & 3 & 5 & 5 & 5 & 3 & 5\\ 0 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 0\\ 5 & 3 & 5 & 3 & 5 & 5 & 5 & 3 & 5\\ 0 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 0\end{pmatrix}.$$

For all $\alpha=\overline{1,2}$ the set of best response strategies of the player 1 is $Br_1(1)=\{1,3\}$, $Br_1(2)=\{1,4\}$, $Br_1(3)=\{1,3\}$, $Br_1(4)=\{1\}$, $Br_1(5)=\{4\}$, $Br_1(6)=\{3\}$, $Br_1(7)=\{2\}$, $Br_1(8)=\{3\}$, $Br_1(9)=\{3\}$. Respectively, for all $\beta=\overline{1,2}$ the set of best response strategies of the player 2 is $Br_2(1)=\{1,3,5,6,7,9\}$, $Br_2(2)=\{2,4,8\}$, $Br_2(3)=\{1,3,5,6,7,9\}$, $Br_2(4)=\{1,3,5,6,7,9\}$. Thus, the set of Bayese-Nash equilibrium profile is

$$BE[\Gamma_{Bayes}]=$$

$$= \{(1_1 1_2, 1_1 1_2), (1_1 1_2, 2_1 1_2), (2_1 1_2, 1_1 1_2), (2_1 1_2, 2_1 1_2), (2_1 1_2, 3_1 3_2)\}.$$

Using given above constructions and the Harsanyi theorem we get the following theorem.

Theorem 6.4.1 *The strategy profile* $(\boldsymbol{i}^*, \boldsymbol{j}^*)$ *is a Bayes-Nash equilibrium in the game* Γ_{Bayes} *from* (6.4.8) *if and only if, for all* $\alpha \in \Delta_1, \beta \in \Delta_2$, *the strategy profile* $(\boldsymbol{i}^*, \boldsymbol{j}^*)$ *is a Nash equilibrium for the subgame* $sub\Gamma_{Bayes} = \langle \{1,2\}, \tilde{\boldsymbol{I}}(\alpha), \tilde{\boldsymbol{J}}(\beta), \boldsymbol{A}(\alpha), \boldsymbol{B}(\beta) \rangle$.

Using the terms of the informational extended strategies, these theorem means the following.

Remark 6.1.2 *If the player* 1 *chooses the information extended strategy* $\theta_1^\alpha \in \Theta_1$ *(respectively, the player* 2 *choose the information extended strategy* $\theta_2^\beta \in \Theta_2$*) and assumes that the player* 2*, for all* $\beta \in \Delta_2$*, will choose the information extended strategies* θ_2^β *with the probability* $p\left(\theta_2^\beta | \theta_1^\alpha\right)$ *(respectively, the player* 2 *assumes that for all* $\alpha \in \Delta_1$*, the player* 1 *will choose the information extended strategies* θ_1^α *with the probability* $q\left(\theta_1^\alpha | \theta_2^\beta\right)$*), then the Nash equilibrium profiles of the bimatrix Bayesian game with matrices* $\boldsymbol{A}(\alpha), \boldsymbol{B}(\beta)$*, for all* $\alpha \in \Delta_1$*,* $\beta \in \Delta_2$*, from (6.4.6)-(6.4.7) is the Bayes-Nash equilibria of the bimatrix informational extended game* $\tilde{\Gamma}$ *from (6.4.5).*

Finally, to determine Bayes-Nash equilibria profiles of the bimatrix incomplete information game

$$\tilde{\boldsymbol{\Gamma}} = \left\langle \{1,2\}, I, J, \left\{ AB(\alpha,\beta) = \left\| \left(a_{ij}^{\alpha\beta}, b_{ij}^{\alpha\beta} \right) \right\|_{i \in I}^{j \in J} \right\}_{\alpha=\overline{1,\kappa_1}}^{\beta=\overline{1,\kappa_2}} \right\rangle$$

from (6.4.5), we have to follow next steps:

• using the "combinatorial algorithm", we construct, for all α, β, the corteges $\mathcal{I}^\alpha$ and $\mathcal{J}^\beta$ that represent the informational extended strategies θ_1^α and θ_2^β, respectively;

• construct the game of incomplete information on the set of information non extended strategies, i.e. construct, for each player, the set of possible utility matrices

$$\left\{ A(\alpha) = ||a_{i_j^\alpha j_i^\beta}||_{i \in I}^{j \in J}, B(\beta) = ||b_{i_j^\alpha j_i^\beta}||_{i \in I}^{j \in J}, i_j^\alpha \in I^\alpha, j_i^\beta \in J^\beta \right\}_{\alpha=\overline{1,\chi_1}}^{\beta=\overline{1,\chi_2}};$$

• for all $\alpha \in \Delta_1$, $\beta \in \Delta_2$, construct the "belief probabilities" $p\left(\theta_2^\beta | \theta_1^\alpha\right)$ and $q\left(\theta_1^\alpha | \theta_2^\beta\right)$;

• generate the sets $\{\tilde{\mathbf{I}}(\alpha)\}_{\alpha \in \Delta_1}, \{\tilde{\mathbf{J}}(\beta)\}_{\beta \in \Delta_2}$ of pure strategies for Bayesian game which correspond to the game $\tilde{\boldsymbol{\Gamma}}$;

• for all fixed $\alpha \in \Delta_1$ and $\beta \in \Delta_2$, construct the payoff matrices $\mathbf{A}(\alpha)$ from (6.4.6) and $\mathbf{B}(\beta)$ *from* (6.4.7);

• using the existent algorithms, determine for all $\alpha \in \Delta_1$, $\beta \in \Delta_2$ the set of Nash equilibrium profiles in the bimatrix game $\langle \{1,2\}, \tilde{\mathbf{I}}(\alpha), \tilde{\mathbf{J}}(\beta), \mathbf{A}(\alpha), \mathbf{B}(\beta) \rangle$.

• using the theorem 6.4.1,construct the set of all Bayes-Nash equilibria in the game Γ_{Bayes} from (6.4.8).

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