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# TRANSPORT PROBLEM ON SUBSURFACES OF MINKOWSKI-COHN SURFACE

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*The transport problem on a convex bounded subsurface with a boundary located on the Minkowski-Cohn surface is posed and studied as an extension of the network transport problem by N.Z. Shor. This extension consists in taking into account and studying the geometry of the transport problem. This allows not only to postulate the distances between the points of the transport network axiomatically, but also to calculate the distances between the points of the network based on the actual geometry of the network. Directions for further research are indicated.*
***Keywords:*** *network transport problem, moduli space, Minkowski-Cohn surface, nonlinearity, distance, optimization, r-algorithm.*

## 1. Introduction

In the work ”Application of the gradient descent method for solving a network transport problem” published for the first time in 1962 (see also [1]]), N.Z. Shor proposed a method for solving a network transport problem based on the ideas of sequential analysis of variants and a variation of the gradient descent method in space of potentials.

The subject of transport problems continues to develop actively [2].

In the future, N.Z. Shor and colleagues have developed more efficient optimization methods and algorithms [3, 4, 5], which have found application in solving transport problems.

In this paper transport problem on a convex bounded subsurface with a boundary located on the Minkowski-Cohn surface is posed and studied as an extension of the network transport problem by N.Z. Shor.

This extension consists in taking into account and studying the geometry of the transport problem. This allows not only to postulate the distances between the points of the transport network axiomatically, but also to calculate the distances between the points of the network based on the actual geometry of the network. The Second section concerns with the Minkowski-Cohn surface and its subsurfaces. As the Minkowski-Cohn surface is the moduli space of admissible lattices the section includes also a sketch of results about algebraic and analytic moduli spaces. Section three present results on computation of distances on the Minkowski-Cohn subsurface. In the conclusion, directions for further research are indicated.

## 2. On the Minkowski-Cohn surface and its subsurfaces

The Minkowski-Cohn surface is the moduli space of admissible lattices of the domain $|x|^p + |y|^p < 1$, $p > 1$, each having 3 pairs of points on the boundary of this domain. Its existence is (implicitly) implied in H. Minkowski's monograph [7] and its explicit construction is given by the Cohn [8] parametrization. In this connection, we recall the notion of a moduli space. Let us first give a brief definition of the moduli space of a class of algebraic objects.

What is moduil in the algebraic case? Classically Riemann claimed that $6g-6$ (real) parameters could be for Riemann surface of genus $g > 1$ which would determine its conformal structure (for elliptic curves, when $g = 1$, it is needs one parameter).

From algebraic point of view we have the following problem: given some kind of variety, classify the set of all varieties having something in common with the given one (same numerical invariants of some kind, belonging to a common algebraic family). For instance, for an elliptic curve the invariant is the modular invariant of the elliptic curve.

Let $\mathbf{B}$ be a class of objects. Let $S$ be a scheme. A family of objects parametrized by the $S$ is the set of objects $X_s : s \in S, X_s \in \mathbf{B}$ equipped with an additional structure compatible with the structure of the base $S$. Algebraic moduli spaces are defined in the papers by Mumford, Harris and Morrison [11, 12].

In the case of analytical parametrization, the situation is similar. Let's present this in our case. At first recall some definitions. Let $M$ be an arbitrary set in $\mathbb{R}^n$, $O=(0,\ldots,0)\in\mathbb{R}^n$. A lattice $\Lambda$ is called admissible for $M$, or $M-$ *admissible*, if it has no points $\neq O$ in the interior of $M$. It is called *strictly admissible* for $M$ if it does not contain a point $\neq O$ of $M$. The *critical determinant* of a set $M$ is the quantity $\Delta(M)$ given by

$$\Delta(M)=inf\left\{d(\Lambda):\Lambda\ strictly\ admissible\ for\ M\right\}$$

with the understanding that $\Delta(M)=\infty$ if there are no strictly admissible lattices. The set $M$ is said to be of the finite or the infinity type according to whether $\Delta(M)$ is finite or infinite.

The moduli space is defined by the equation

$$\Delta(p,\sigma)=(\tau+\sigma)\left(1+\tau^p\right)^{-\frac{1}{p}}\left(1+\sigma^p\right)^{-\frac{1}{p}}, \tag{1}$$

in the domain

$$\mathcal{M}:\infty>p>1,\ 1\leq\sigma\leq\sigma_p=\left(2^p-1\right)^{\frac{1}{p}},$$

of the $\{p,\sigma\}-$plane, where $\sigma$ is some real parameter; here $\tau=\tau(p,\sigma)$ is the function uniquely determined by the conditions

$$A^p+B^p=1,\ 0\leq\tau\leq\tau_p,$$

where

$$A=A(p,\sigma)=\left(1+\tau^p\right)^{-\frac{1}{p}}-\left(1+\sigma^p\right)^{-\frac{1}{p}},$$

$$B=B(p,\sigma)=\sigma\left(1+\sigma^p\right)^{-\frac{1}{p}}+\left(1+\tau^p\right)^{-\frac{1}{p}},$$

$\tau^p$ is defined by the equation $2\left(1-\tau_p\right)^p=1+\tau_p^p,\ 0\leq\tau_p<1$.

**Definition 1.** In the notation above, the surface

$$\Delta-(\tau+\sigma)\left(1+\tau^p\right)^{-1/p}\left(1+\sigma^p\right)^{-1/p}=0,$$

in $\mathbb{R}^3$ with coordinates (parameterization) $(\sigma,p,\Delta)$ we will called the *Minkowski-Cohn moduli space*, or shortly the *Minkowski-Cohn surface*. We will denote this surface as $\mathcal{MC} = \mathcal{MC}(\sigma, p, \Delta)$. The $\Delta$ parameter is explicitly represented by parameters $(\sigma, p)$ so we will denote this as $\mathcal{MC} = \mathcal{MC}(\sigma, p)$. We will using similar notations for subsurfaces of the Minkowski-Cohn surface.

Let $P_a = \{p = a \in \mathbb{R},\ a > 1\}$ be the plane in $\mathbb{R}^3$ with coordinates $(\sigma, p, \Delta)$.

*Remark 1*. The intersection $\mathcal{MC} \cap P_a$ defines the bounded Minkowski-Cohn subsurface $\mathcal{MC}_a$

$$\Delta - (\tau + \sigma)(1 + \tau^p)^{-1/p}(1 + \sigma^p)^{-1/p} = 0,$$

in the domain

$$\mathcal{M}_a : 1 < p \le a,\ 1 \le \sigma \le \sigma_p = (2^p - 1)^{\frac{1}{p}}.$$

*Remark 2*. Respectively the intersection $\mathcal{MC} \cap P_a$ and $\mathcal{MC} \cap P_b$, $a < b \in \mathbb{R}$ defines the bounded Minkowski-Cohn subsurface $\mathcal{MC}_{a,b}$

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in the domain

$$\mathcal{M}_{a,b} : a \le p \le b,\ 1 \le \sigma \le \sigma_p = (2^p - 1)^{\frac{1}{p}}.$$

## 3. On distances on the Minkowski-Cohn subsurface

As we will consider the transport problem on the Minkowski-Cohn subsurface $\mathcal{MC}_{a,b}$ we need the notion of the distance. Recall the main result of the proof of the Minkowski conjecture. In notations [6, 9, 10] next result have proved:

**Theorem 1.** [10]

$$\Delta(D_p) = \begin{cases} \Delta(p,1), & 1 < p \le 2,\ p \ge p_0, \\ \Delta(p,\sigma_p), & 2 < p \le p; \end{cases}$$

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**Lemma 2.** *In notations of the Definition (2) the Riemannian metric on the surface* $\mathcal{M}_{a,b}$ *has the form*

$$ds^2 = \left(\Delta'_\sigma\right)^2 d\sigma^2 + 2\Delta'_\sigma \Delta'_p d\sigma dp + \left(\Delta'_p\right)^2 dp^2 .$$

*The length* $l(M,N)$ *of an arc of the curve on the surface* $\mathcal{M}_{a,b}$*, joining the points* $M = \Delta\left(\sigma\left(t_0\right), p\left(t_0\right)\right)$ *and* $N = \Delta\left(\sigma\left(t_1\right), p\left(t_1\right)\right)$ *is equal to*

$$l\left(M,N\right) = \int_{t_0}^{t_1} \sqrt{\left(\Delta'_\sigma\right)^2 d\dot{\sigma}^2 + 2\Delta'_\sigma \Delta'_p d\dot{\sigma} d\dot{p} + \left(\Delta'_p\right)^2 d\dot{p}^2}\, dt.$$

We continue use notations of the Definition (2) and introduce Gauss notations: $E = \left(\Delta'_\sigma\right)^2$, $F = \Delta'_\sigma \Delta'_p$, $G = \left(\Delta'_p\right)^2$. Let $M = \left(\sigma, p\right) \in \Delta$ be a point with tangent space at $M$. Using Gauss notations, put $Q\left(x, y\right)$ for quadratic form at $M$:

$$Q_M\left(\sigma, p\right) = \left(\sigma, p\right) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \sigma \\ p \end{pmatrix}.$$

**Lemma 3.** *Quadratic form* $Q_M\left(\sigma, p\right)$ *defines the symmetric bilinear form* $B_M\left(\left(\sigma_1, p_1\right), \left(\sigma_2, p_2\right)\right)$ *which is an inner product* $\left\langle \left(\sigma_1, p_1\right), \left(\sigma_2, p_2\right) \right\rangle$ *on the tangent space at* $M$*:*

$$\left\langle \left(\sigma_1, p_1\right), \left(\sigma_2, p_2\right) \right\rangle = B_M\left(\left(\sigma_1, p_1\right), \left(\sigma_2, p_2\right)\right) = \left(\sigma_1, p_1\right) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \sigma_2 \\ p_2 \end{pmatrix}.$$

By the scaler product it is possible to define the angle between of two curves passing through $M \in \mathcal{M}_{a,b}$. This is the angle $\alpha$ of the tangent vectors to these two curves at $M$:

$$\alpha = \arccos\left( \frac{\left\langle \left(\dot{\sigma}_1, \dot{p}_1\right), \left(\dot{\sigma}_2, \dot{p}_2\right) \right\rangle}{\sqrt{Q\left(\dot{\sigma}_1, \dot{p}_1\right)} \sqrt{Q\left(\dot{\sigma}_2, \dot{p}_2\right)}} \right).$$

Functions $\frac{\partial\Delta}{\partial\sigma}=\Delta'_\sigma$, $\frac{\partial\Delta}{\partial p}=\Delta'_p$, $\frac{\partial^2\Delta}{\partial\sigma^2}=\Delta''_{\sigma^2}$, $\frac{\partial^2\Delta}{\partial p^2}=\Delta''_{p^2}$, $\frac{\partial^2\Delta}{\partial\sigma\partial p}=\Delta''_{\sigma p}$ is represented in terms of a sum of derivatives of ”atoms” $s_i=\sigma^{p-i}$, $t_i=\tau^{p-i}$, $a_i=\left(1+\sigma^p\right)^{-i-\frac{1}{p}}$, $b_i=\left(1+\tau^p\right)^{-i-\frac{1}{p}}$, $A=b_0-a_0$, $B=\tau b_0+\sigma a_0$, $\alpha_i=A^{p-i}$, $\beta_i=B^{p-i}$ $(i=0,1,2,\ldots)$.

Then by the implicit function theorem computing $\tau=\tau(p,\sigma)$ by means of the following iteration process:

$$\tau_{i+1}=\left(1+\tau_i^p\right)^{\frac{1}{p}}\left(\left(1-\left(\left(1+\tau_i^p\right)^{-\frac{1}{p}}-\left(1+\sigma^p\right)^{-\frac{1}{p}}\right)^p\right)^{1/p}-\sigma\left(1+\sigma^p\right)^{-\frac{1}{p}}\right),$$

For computation of the expression for $\tau_p$ we apply the following iteration:

$$\left(\tau_p\right)_{i+1}=1-\left(2^{-\frac{1}{p}}\right)\left(1+\left(\tau_p\right)_i^p\right)^{\frac{1}{p}},\ p>1,\ \left(\tau_p\right)_0\in[0,0.36].$$

Under given $p$ with increasing $\sigma$ from 1 to $\sigma_p$ the function $\tau=\tau(p,\sigma)$ is strictly monotonically decreasing from $\tau_p$ to 0; $\Delta(p,\ 1)=\Delta_p^{(1)}=4^{-\frac{1}{p}}\frac{1+\tau_p}{1-\tau_p}$, $\Delta(p,\ \sigma_p)=\Delta_p^{(0)}=\frac{1}{2}\sigma_p$.

**Lemma 4.** $\frac{\partial\Delta(p,\sigma)}{\partial\sigma}\big|_{\sigma=1}=\frac{\partial\Delta(p,\sigma)}{\partial\sigma}\big|_{\sigma=\sigma_p}=0.$

## 4. Conclusions

We extend the framework of the network transport problem by N.Z. Shor in the direction of the geometry of the transport problem in the case when the transport problem is posted on the sub surface of the Minkowski-Cohn surface. The class of the Minkowski-Cohn surfaces of the form $\mathcal{M}_{a,b}$ has introduced and studied. Results about finding of distances on the Minkowski-Cohn sub surface are presented. We will plan to continue investigations of the Shor's network transport problem on the base of the geometry of the problem and on the base of some novel optimization algorithms.

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