The Independence Number of the Generalized Wheel Graphs W_{2k+1}^p

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CHAPTER 6

The Independence Number of the
Generalized Wheel Graphs W^{p}_{2k+1}

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1. Introduction

Combinatorial Optimization Problems (COPS) are ubiquitous and widespread in modeling of real-world problems of Geometric Design and Operational Research (see, for instance, Pardalos et al. 2013).

A huge contribution to Combinatorial Optimization and Computational Complexity Theory was made by N. Z. Shor. He proposed an original technique for obtaining dual bounds on the optimal value of the objective function in quadratic optimization problems (Shor 2011). This technique includes an algorithm for obtaining the quadratic dual bounds by solving auxiliary non-smooth continuous optimization problems called the r-algorithm (see Shor et al. 1997). Another component of the approach is searching for superflows constraints for the initial problem and then utilizing them to refine the dual bounds. The additional constraints depend on the type of problems and are selected such that the above bounds can be improved.

$\mathbb{B}^n$ analytic representations

$$\mathbb{B}^n = \{0, 1\}^n$$

Equivalent analytic representation of the condition $x \in \mathbb{B}^n$:

$$^*x_i^2 - x_i = 0, i = \overline{1, n}$$
$$(x_1 - 0.5)^2 + ... + (x_n - 0.5)^2 = n/4; 0 \le x_i \le 1, i = \overline{1, n}$$

$$(x_1 - 0.5)^2 + ... + (x_n - 0.5)^2 = n/4;$$
$$(x_1 - 0.5)^4 + ... + (x_n - 0.5)^4 = n/16$$

The maximum independent set problem (MIS)

Let $G = (V(G), E(G))$ be an undirected graph with vertex set $V(G) = \{1, ..., n\}$ and edge set $E(G) \subseteq V(G) \times V(G)$.

An **independent set** (stable set) in G is a subset of $V(G)$ (further $S \subseteq V(G)$), whose elements are pairwise non-adjacent.

Let $\mathcal{S}(G)$ be a set of all independent sets in G. Then

$$\forall S \in \mathcal{S}(G), \forall i, j \in S, i \neq j, \{i, j\} \notin E(G). \tag{1}$$

MIS aims to find an independent set $S^* \in \mathcal{S}(G)$ of maximum cardinality. The size of a maximum independent set is called the **stability number** of G and is denoted by $\alpha(G)$.

$$\alpha(G) = |S^*| = \max_{S \in \mathcal{S}(G)} |S|. \tag{2}$$

COPs and ECOPs

Combinatorial Optimization Problem (**COP**): find

$$f^* = f(\pi^*) = \operatorname*{extr}_{\pi \in \Pi' \subseteq \Pi} f(\pi), \tag{3}$$

where Π is a search domain, which is a combinatorial set, such as the permutation set, sets of graphs, and Π' is a feasible domain.

Euclidean Combinatorial Optimization Problems (**ECOP**): find

$$\varphi^* = \varphi(x^*) = \operatorname*{extr}_{x \in X' \subseteq X} \varphi(x), \tag{4}$$

where X' and X are feasible and search domains, respectively,

$$X' \subseteq X \subseteq \mathbb{R}^n$$

A COP and an ECOP are equivalent if

$$\varphi^* = f^*, \ \exists \chi : \pi^* = \chi^{-1}(x^*), \ \text{where } x^* = \operatorname*{argextr}_{x \in X' \subseteq X} \varphi(x).$$

MIS: COP and ECOPs

MIS.COP: find $\langle S^*, \alpha(G) \rangle$

$$\alpha(G) = |S^*| = \max_{S \in \mathcal{S}(G)} |S|. \quad (2)$$

$$\forall S \in \mathcal{S}(G), \forall i, j \in S, i \neq j, \{i, j\} \notin E(G). \quad (1)$$

MIS.ECOPs: If $\pi \subseteq V = (v_i)_{i=\overline{1,n}}$, a standard way to construct an ECOP associated with such a COP is considering incidence vectors corresponding to a set π:

$$\chi : \pi \to \chi(\pi) = \{x \in \mathbb{B}^n : x_i = 1, \text{ if } v_i \in \pi; \ 0, \text{ otherwise}\}. \quad (5)$$

Let $x = (x_1, .., x_n)$ be a vector of variables. MISP aims to find:

$$\alpha(G) = e^\top x^* = \max \sum_{i \in V(G)} x_i = \max e^\top x$$

s.t. x is an image of an independent set $S \in \mathcal{S}(G)$ under mapping χ given by (5)

This MIS formulation is still a COP, not ECOP, due to the combinatorial constraint $S \in \mathcal{S}(G)$

MIS: ECOPs (cont.), analytic representations of (6)

· **vertex constraint (VC)**:

$$(\text{binary VC, BVC}) : x_i \in \{0,1\}, i \in V(G),$$
$$(\text{continuous VC, CVC}) : x_i^2 - x_i = 0, i \in V(G),$$
$$(\text{continuous relaxation of BVC, RVC}) : 0 \le x_i \le 1, i \in V(G).$$

· **edge constraint (EC)**:

$$(\text{binary EC, BEC}) : \quad x_i + x_j \in \{0,1\}, \{i,j\} \in E(G),$$
$$(\text{continuous EC, CEC}) : \quad (x_i + x_j)^2 - (x_i + x_j) = 0, \{i,j\} \in E(G),$$
$$(\text{continuous relaxation of BEC, REC}) : \quad 0 \le x_i + x_j \le 1, \{i,j\} \in E(G),$$
$$(\text{linear EC, LEC}) : \quad x_i + x_j \le 1, \{i,j\} \in E(G).$$

· **clique constraint (CC)**: Let $\mathcal{Q}(G)$ be a clique set in G.

$$(\text{binary CC, BCC}) : \quad \sum_{i \in Q} x_i \in \{0,1\}, Q \in \mathcal{Q}(G),$$
$$(\text{continuous CC, CCC}) : \quad (\sum_{i \in Q} x_i)^2 - (\sum_{i \in Q} x_i) = 0, Q \in \mathcal{Q}(G),$$
$$(\text{continuous relaxation of BCC, RCC}) : \quad 0 \le \sum_{i \in Q} x_i \le 1, Q \in \mathcal{Q}(G),$$
$$(\text{linear CC, LCC}) : \quad \sum_{i \in Q} x_i \le 1, Q \in \mathcal{Q}(G).$$

By CVC, CEC and CCC are simplified:

$$\forall \{i,j\} \in E(G) \quad (x_i + x_j)^2 - (x_i + x_j) =$$
$$= 2x_ix_j + (x_i^2 - x_i) + (x_j^2 - x_j) \underset{CVC}{=} 2 \cdot x_ix_j = 0.$$
$$\forall Q \in \mathcal{Q}(G) \quad (\sum_{i \in V(Q)} x_i)^2 - \sum_{i \in V(Q)} x_i =$$
$$= 2 \cdot \sum_{i<j, \{i,j\} \in V(Q)} x_ix_j + \sum_{i \in V(Q)} (x_i^2 - x_i) \underset{CVC}{=} 2 \cdot \sum_{i<j, \{i,j\} \in V(Q)} x_ix_j = 0.$$

(quadratic EC, QEC) : $x_ix_j = 0, \{i,j\} \in E(G)$
(quadratic CC, QCC) : $\sum_{i<j, \{i,j\} \in Q} x_ix_j = 0, Q \in \mathcal{Q}(G)$

(6)+*VC+**C is a MIS.ECOP, where $* = \{B, C\}$,
$** = \{BE, CE, QE, RE, LE; BC, CC, QC, RC, LC\}$

The polytope STAB(G) and its relaxations

A feasible domain of a MIS.ECOP:

$$X' = \{x \in X : \pi = \chi^{-1}(x) \in \mathcal{S}(G)\}.$$

(6) s.t. $x \in X'$ is a MIS formulation. Its polyhedral relaxation is (6) s.t.

$$x \in P' = \text{conv}(X').$$

Conventionally, the polytope P' is denoted as $STAB(G)$, thus

$$STAB(G) = \text{conv}\{\chi(S) : S \in \mathcal{S}(G)\}.$$

Let

$$\mathcal{L}STAB(G) = \{x \in \mathbb{R}^n : x \text{ satisfies linear constraints } \mathcal{C}(\mathcal{L})\},$$
$$STAB(G) \subseteq \mathcal{L}STAB(G),$$

i.e., $\mathcal{L}STAB(G)$ is a polyhedral outer approximation of $STAB(G)$.

$\mathcal{G}(\mathcal{L})$ graphs

If for a graph G

$$\mathcal{L}STAB(G) = STAB(G), \tag{7}$$

then the graph G is called $\mathcal{G}(\mathcal{L})$ graph.

Certain combinations of $\mathcal{L}$ and $\mathcal{G}$ are listed in two-line notation in Table 1.

Table: Correspondence of $\mathcal{L}$-polytopes to $\mathcal{G}(\mathcal{L})$-types of graphs

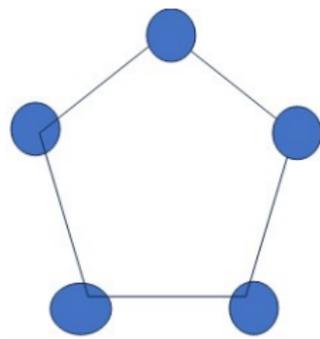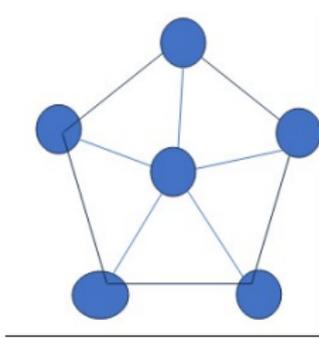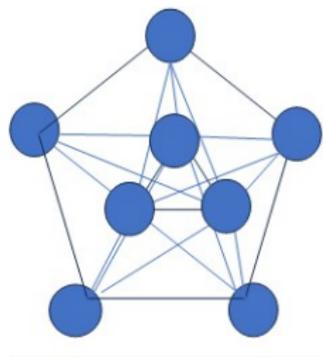| $\mathcal{L}$	F	Q	C	H	W_{2k+1}^1	W_{2k+1}^p
$\mathcal{G}(\mathcal{L})$	bipartite	perfect	t-perfect	h-perfect	w_{2k+1}^1-perfect	w_{2k+1}^p-perfect

Figure: C_5

Figure: W_5^1

Figure: W_5^3

Let

$$\alpha_{\mathcal{L}}^*(G) = \max_{x \in \mathcal{L}STAB(G)} \sum_{i \in V(G)} x_i$$

For every $\mathcal{L}STAB(G)$,

$$\alpha_{\mathcal{L}}^*(G) \geq \alpha(G) \tag{8}$$

If $\mathcal{L} \in \{F, Q, C, H, W_{2k+1}^1, W_{2k+1}^p\}$, then, by (7), the relation (8) becomes

$$\alpha_{\mathcal{L}}^*(G) = \alpha(G) \tag{9}$$

FSTAB(G), QSTAB(G)

· fractional polytope:

$\mathcal{L} = F, FSTAB(G) = \{x \in \mathbb{R}^n : x$ satisfies constraints RVC, LEC$\}$

· clique polytope:

$\mathcal{L} = Q, QSTAB(G) = \{x \in \mathbb{R}^n : x$ satisfies constraints RVC, LCC$\}$

(7) provides an alternative definition of graphs presented in Table 1, e.g.

Definition 1.1.

If for G,

- $FSTAB(G) = STAB(G)$, then G is bipartite;
- $QSTAB(G) = STAB(G)$, then G is perfect.

CSTAB(G), HSTAB(G)

Other MIS valid linear constraints: **odd-cycle constraints (OCC)**

$$(OCC): \quad \sum_{i \in V(C_{2k+1})} x_i \le k \quad \forall k \le [n/2], \forall C_{2k+1} \in \mathcal{C}(G),$$

where C_{2k+1} is an odd cycle in graph G, $\mathcal{C}(G)$ is a set of odd cycles in G.

· C-polytope:

$$\mathcal{L} = C, CSTAB(G) = \{x \in \mathbb{R}^n : x \text{ satisfies constraints RVC, LEC, OCC}\}$$

· H-polytope:

$$\mathcal{L} = H, HSTAB(G) = \{x \in \mathbb{R}^n : x \text{ satisfies constraints RVC, RCC, OCC}\};$$

Definition 1.2.

If for G,

- $CSTAB(G) = STAB(G)$, then G is t-perfect;
- $HSTAB(G) = STAB(G)$, then G is h-perfect.

Odd-wheel and generalized odd-wheel constraints

Other MIS valid linear constraints: **odd-wheel constraints (OWC)**

$$\text{(OWC)} \qquad \sum_{i \in V(C_{2k+1})} x_i + k x_{i_{2k+2}} \leq k,$$

$$\forall k \leq [(n-1)/2], \forall\, W^1_{2k+1} \in \mathcal{W}^1(G),$$

where W^1_{2k+1} is an odd-wheel subgraph on $2k+2$ vertices in the graph G, $\mathcal{W}^1(G)$ is a set of odd-wheel subgraphs in G.

generalized odd-wheel constraints (GOWC)

(GOWC)

$$\sum_{i \in V(C_{2k+1})} x_i + k \sum_{j \in V(Q_p)} x_j \leq k \qquad \forall p \leq N, \forall k \leq [(n-p)/2], \forall W^p_{2k+1} \in \mathcal{W}(G),$$

where N is a clique number of G, the graph W^p_{2k+1} is the generalized wheel graph, which is a join of an odd circle $C_{2k+1} \in \mathcal{C}(G)$ and a p-clique $Q_p \in \mathcal{Q}(G)$, $\mathcal{W}(G)$ be a set of W^p_{2k+1}-type graphs, i.e., $\mathcal{W}(G) = \{W^p_{2k+1}\}_{k,p}$.

$W_{2k+1}^p STAB(G)$, $p \geq 1$

· W_{2k+1}^1-polytope:

$$\mathcal{L} = W_{2k+1}^1, W_{2k+1}^1 STAB(G) = \{x \in \mathbb{R}^n : x \text{ satisfies } RVC, LEC, OCC,$$
$$OWC \text{ for } W_{2k'+1}^1, k' = \overline{1,k}\}$$

· W_{2k+1}^p-polytope, where $p \geq 2$:

$$\mathcal{L} = W_{2k+1}^p, W_{2k+1}^p STAB(G) = \{x \in \mathbb{R}^n : x \text{ satisfies } RVC, LEC, OCC,$$
$$GOWC \text{ for } W_{2k'+1}^{p'}, k' = \overline{1,k}, p' = \overline{1,p}\}$$

Definition 1.3.

If for G,

- $W_{2k+1}^p STAB(G) = STAB(G)$, then G is W_{2k+1}^p-perfect.

Consider the problem (10), (11) (further **Problem 0**)

$$Q_0^* = Q_0(x^*) = \sup_{x \in \mathbb{R}^n} Q_0(x) \tag{10}$$

subject to constraints

$$\begin{aligned} Q_i(x) &\leq 0, \quad i = 1, \ldots, m; \\ Q_i(x) &= 0, \quad i = m+1, \ldots, M, \end{aligned} \tag{11}$$

where $M \geq 1$, $Q_i(x) = (K_i x, x) + (b_i, x) + c_i$, K_i is a symmetric matrix of the dimension n, $b_i \in \mathbb{R}^n$, $c_i \in \mathbb{R}$, $i = 1, \ldots, M$.

The bound ψ^* is the optimal value of the Lagrangian dual problem to the primal Problem 0, i.e.,

$$\psi^* = \inf_{u \in \mathcal{U}^+} \psi(u), \tag{12}$$

where $u \in \mathcal{U}^+ = \mathbb{R}_+^m \times \mathbb{R}^{M-m}$ is a vector of dual variables,

$$\psi(u) = \sup_{x \in \mathbb{R}^n} L(x, u),$$

$$L(x, u) = Q_0(x) - \sum_{i=1}^{M} u_i Q_i(x)$$

The simple bound ψ^*

If (10), (11) is noncompatible, we set $Q_0^* = -\infty$.

$$L(x, u) = Q_0(x) + \sum_{i=1}^{M} u_i Q_i(x) = (K(u)x, x) + (b(u), x) + c(u),$$

where

$$K(u) = K_0(x) - \sum_{i=1}^{M} u_i K_i(x), \quad b(u) = b_0 - \sum_{i=1}^{M} u_i b_i, \quad c(u) = c_0 - \sum_{i=1}^{M} u_i c_i.$$

The domain $\mathcal{U}^+$ can be partitioned as follows: $\mathcal{U}^+ = \Omega^- \cup \Omega^0 \cup \Omega^+$, where $\Omega^- = \{u : K(u) \prec 0\}$, $\Omega^+ = \{u : K(u) \succ 0\}$, $\Omega^0 = \{u : K(u) \preceq 0\} \backslash \Omega^-$. Depending on the domain where u falls, we have three cases:

- If $u \in \Omega^+$, then $\psi^* = +\infty$, and we set $Q_0^* = \psi^* = +\infty$;
- if $u \in \Omega^-$, then $\psi(u)$ is strictly convex, and solving (12) is reduced to solution of the linear algebraic system;
- if $u \in \Omega^0$, then solving (12) is complicated and may require applying nonsmooth optimization methods.

Shor's dual bounds on Q_0^*: the improved bound ψ_1^*

Let the below constraints are superflows for Problem 0:

$$\begin{aligned}
Q_i(x) &\leq 0, \quad i = M+1, \ldots, M+I; \\
Q_i(x) &= 0, \quad i = M+I, \ldots, M+L,
\end{aligned} \tag{13}$$

where $L \geq 1$, function $Q_i(x)$ is quadratic for $i = M+1, \ldots, M+L$.

We introduce **Problem 1** as Problem 0 complemented by constraints (13).

The bound ψ_1^* is the optimal value of the Lagrangian dual problem to the primal Problem 1, i.e.,

$$\psi_1^* = \inf_{U \in \mathcal{U}_1^+} \psi_1(U), \tag{14}$$

where $U = (u_1, ..., u_{M+L}) \in \mathcal{U}_1^+ = U_+ \times \mathbb{R}_+^I \times \mathbb{R}^{L-I}$ is a vector of dual variables,

$$\psi_1(U) = \sup_{x \in \mathbb{R}^n} L_1(x, U),$$

$$L_1(x, U) = Q_0(x) - \sum_{i=1}^{M+L} u_i Q_i(x) = (K_1(U)x, x) + (b_1(U), x) + c_1(U),$$

$$K_1(U) = K_0(x) - \sum_{i=1}^{M+L} u_i K_i(x), \quad b(U) = b_0 - \sum_{i=1}^{M+L} u_i b_i,$$

$$c(U) = c_0 - \sum_{i=1}^{M+L} u_i c_i.$$

!!! a) $\psi_1^* \leq \psi^*$, b) complexity: $\psi^* - O(n^3) + O(M^2)$, $\psi_1^* - O(n^3) + O((M+L)^2)$

Specifics of constructing superflows constraints

Now the question arises of which superflows constraints can improve the upper bound ψ^* by finding ψ_1^*. Examples are:

- **Type 1**: linear combinations of constraints' components such as $Q_i(x) + Q_j(x) \leq 0$ or $Q_i(x) + Q_j(x) = 0$;

- **Type 2** : utilizing multiplication of constraints' components, e.g., $Q_i(x)Q_j(x) \geq 0$ or $Q_i(x)Q_j(x) = 0$.

Lemma 1.4.

If, for Problem 1, $l = 0$ and

$$\exists \lambda_{ik} \in \mathbb{R}^1, i = \overline{m+1, M}, k = \overline{l+1, L},$$

$$\text{such that } Q_{M+k}(x) = \sum_{i=m+1}^{M} \lambda_{ik} Q_i(x) \text{ for } k = \overline{l+1, L},$$

then

$$\psi_1^* = \psi^*. \tag{15}$$

MIS models utilized in $\alpha_{\mathcal{L}}(G)$, $\psi^*(G)$, $\psi_1^*(G)$

For MIS:

$$Q_0^* \Longrightarrow \alpha(G)$$
$$\psi^* \Longrightarrow \psi^*(G),\ \psi_1^* \Longrightarrow \psi_1^*(G)$$

1. Continuous **quadratic MIS model, MIS.QM**: The objective (6) and constraints CVC + (QEC or QCC)

$$\alpha(G) = \max \sum_{i \in V(G)} x_i \tag{16}$$

$$(\text{CVC}): \quad x_i^2 - x_i = 0, i \in V(G)$$
$$(\text{QEC}): \quad x_i x_j = 0, \{i, j\} \in E(G)$$
$$(\text{QCC}): \quad \sum_{i < j, \{i,j\} \in Q} x_i x_j = 0, Q \in \mathcal{Q}(G).$$

2. **polyhedral relaxation, MIS.PR**: the objective (16) and constraints RVC+(LEC or LCC).

$$(\text{RVC}): \quad 0 \le x_i \le 1, i \in V(G)$$
$$(\text{LEC}): \quad x_i + x_j \le 1, \{i, j\} \in E(G)$$
$$(\text{LCC}): \quad \sum_{i \in Q} x_i \le 1, Q \in \mathcal{Q}(G)$$

$$f_{ij} = x_i + x_j - 1 \leq 0, \{i, j\} \in E(G);$$
$$f_k' = -x_k \leq 0, k = \overline{1, n};$$
$$f_k'' = x_k - 1 \leq 0, k = \overline{1, n}.$$

$$f_{ij} f_{i'j'} = (x_i + x_j - 1)(x_{i'} + x_{j'} - 1) \geq 0, \{i, j\}, \{i', j'\} \in E(G);$$
$$f_k' \cdot f_{k'}' = (-x_k)(-x_{k'}) \geq 0, k, k' = \overline{1, n}$$
$$f_k'' \cdot f_{k'}'' = (x_k - 1)(x_{k'} - 1) \geq 0, k, k' = \overline{1, n}$$
$$f_{ij} \cdot f_k' = (x_i + x_j - 1)(-x_k) \geq 0, \{i, j\} \in E(G), k = \overline{1, n}$$
$$f_{ij} \cdot f_k'' = (x_i + x_j - 1)(x_k - 1) \geq 0, \{i, j\} \in E(G), k = \overline{1, n}$$
$$f_{k'}' \cdot f_{k''}'' = (-x_{k'})(x_{k''} - 1) \geq 0, k', k'' = \overline{1, n}$$

$$f_{ij} \cdot f_k' \geq 0 \Longrightarrow$$
(superflows constraints, SC) $x_k(x_i + x_j) \leq x_k, \{i, j\} \in E(G), k = \overline{1, n}.$

$\psi^*(G)$-application

The idea to prove polynomiality of obtaining $\alpha(G)$ for graphs listed in Table 1 is based on justifying that the corresponding constraints of MIS.PR follows from the quadratic constraints of MIS.QM, wherefrom

$$\psi^*(G) \leq \alpha^*_{\mathcal{L}}(G).$$

It is justified for $\mathcal{L} \in \{F, Q\}$. Together with $\psi^*(G) \geq \alpha(G)$ and (18), it yields

$$\alpha^*_{\mathcal{L}}(G) = \psi^*(G) = \alpha(G) \tag{17}$$

for bipartite and perfect graphs.

$\psi_1^*(G)$-application

For the rest families of graphs, the polynomiality was established by utilizing $\psi_1^*(G)$. For that, MIS.PR was transformed into MIS.PR+ by complementing it by OCC, OWC or GOWC, while MIS.QM was transformed into MIS.QM+ by its complementing by SC. The above idea now becomes: to establish that the corresponding constraints of MIS.PR+ follows from quadratic constraints of MIS.QM+, then

$$\psi_1^*(G) \leq \alpha_{\mathcal{L}}^*(G).$$

It was justified for $\mathcal{L} \in \{C, H, W_{2k+1}^1, W_{2k+1}^p\}$. Combining it with $\psi_1^*(G) \geq \alpha(G)$ and (18), we come to

$$\alpha_{\mathcal{L}}^*(G) = \psi_1^*(G) = \alpha(G) \tag{18}$$

for t-perfect graphs, h-perfect graphs, wheel-perfect graphs and generalized wheel-perfect graphs.

Thank you!
Questions?