Subgradient method with Polyak's step in transformed space

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Outline

1. Polyak's subgradient method ($m = 1$)
2. Method A ($m \geqslant 1$)
3. Method B ($m \geqslant 1$, matrix B)
4. Comparison of the methods A and B

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Formulation of the problem

We consider the following problem

$$\text{to find} \quad x^\star = \arg\min_{x \in R^n} f(x) \quad \text{if } f^\star \text{ is known,} \tag{1}$$

where $f(x)$ is a convex function and $f^\star = f(x^\star) = \min_{x \in R^n} f(x)$.

Main inequality for the subgradient

For convex function $f(x)$ the following inequality is valid

$$(x - x^\star, g_f(x)) \geq f(x) - f^\star, \quad \forall x \in R^n, \tag{2}$$

where $g_f(x)$ is a subgradient (gradient) of the function $f(x)$.

Inequality (2) is used for the calculation of the step in subgradient method, which B.T. Polyak offered in 1969.

Polyak B.T. Minimization of unsmooth functionals // *USSR Comput. Math. Math. Phys.*. 1969. Vol. 9, No. 3, pp. 14-29.

Polyak's subgradient method

Polyak's subgradient method has the iterative form

$$x_{k+1} = x_k - h_k \frac{g_f(x_k)}{\|g_f(x_k)\|}, \quad h_k = \frac{f(x_k) - f^\star}{\|g_f(x_k)\|}, \quad k{=}0, 1, \ldots \tag{3}$$

Step h_k is called Polyak's step.

Decrease of the distance to the minimum point

Theorem 1 (Polyak, 1969)

The sequence $\{x_k\}_{k=0}^{\infty}$, *generated by the method (3), satisfies the inequalities*

$$\|x_{k+1} - x^\star\|^2 \leq \|x_k - x^\star\|^2 - \frac{(f(x_k) - f^\star)^2}{\|g_f(x_k)\|^2}, \quad k = 0, 1, 2, \ldots$$

Remark. Theorem 1 guarantees that in Polyak's method the distance to the minimum point decreases monotonically.

m-inequality for the subgradient

We consider methods A and B for convex functions $f(x)$, subgradients $g_f(x)$ of which satisfy the following condition:

$$(x - x^\star, g_f(x)) \geq m(f(x) - f^\star), \quad \forall x \in R^n, \tag{4}$$

where parameter $m \geq 1$.

The method A uses the step h_k in **original space**, the method B uses h_k in **transformed space** of variables.

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Method A

Initialization. $f^\star$, $m \geq 1$, $x_0 \in R^n$, $\varepsilon > 0$.

A1. Calculate $f(x_k)$ and $g_f(x_k)$.
If $f(x_k) - f^* \leq \varepsilon$, then STOP ($k^* = k$, $x_\varepsilon^* = x_k$).

A2. Calculate the next point

$$x_{k+1} = x_k - h_k \frac{g_f(x_k)}{\|g_f(x_k)\|}, \quad h_k = \frac{m(f(x_k) - f^*)}{\|g_f(x_k)\|}.$$

A3. Go to the $(k+1)$-th iteration with x_{k+1}.

Decrease of the distance to the minimum point

Theorem 2 (Stetsyuk, 2012)

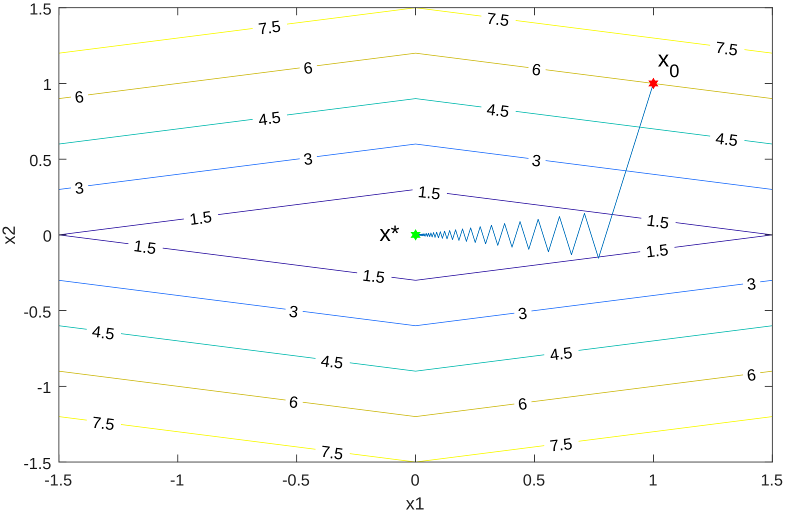
The sequence $\{x_k\}_{k=0}^{k^*-1}$*, generated by the method A, satisfies the inequalities*

$$\|x_{k+1} - x^\star\|^2 \leq \|x_k - x^\star\|^2 - \frac{m^2(f(x_k) - f^\star)^2}{\|g_f(x_k)\|^2}, \quad k = 0, 1, 2, \ldots \quad (5)$$

Remark. Theorem 2 guarantees that in Polyak's subgradient method the distance to the minimum point decreases monotonically if the inequality (4) is used.

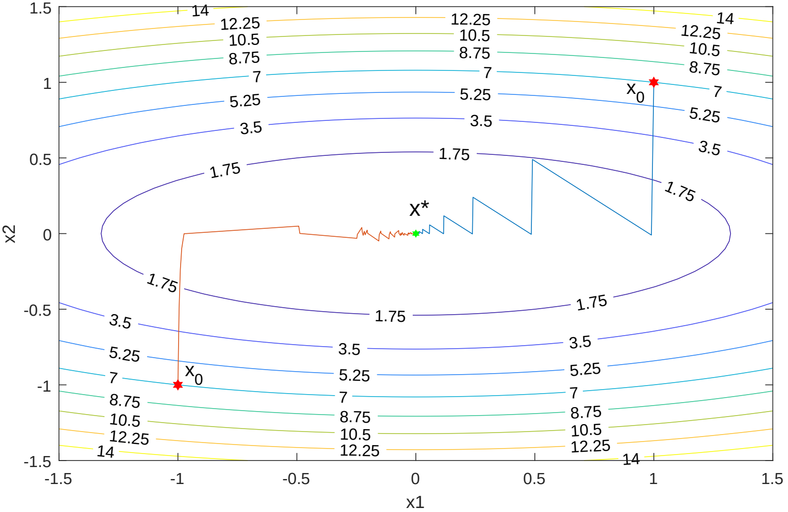
Method A for $f_1(x_1, x_2) = |x_1| + 5|x_2|$

$$m = 1, \ f^\star = 0, \ x_0 = (1, 1)^T$$



Method A for $f_2(x_1, x_2) = x_1^2 + 6x_2^2$

Blue - $m = 2,\ x_0 = (1, 1)^T$ Red - $m = 1,\ x_0 = (-1, -1)^T$



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Method B

Initialization. $f^{\star}$, $m \geq 1$, $x_0 \in R^n$, $n \times n$ matrix B, $\varepsilon > 0$.

A1. Calculate $f(x_k)$ and $g_f(x_k)$.
If $f(x_k) - f^* \leq \varepsilon$, then STOP ($k^* = k$, $x_\varepsilon^* = x_k$).

A2. Calculate the next point

$$x_{k+1} = x_k - h_k B \frac{B^T g_f(x_k)}{\|B^T g_f(x_k)\|}, \quad h_k = \frac{m(f(x_k) - f^*)}{\|B^T g_f(x_k)\|}.$$

A3. Go to the $(k+1)$-th iteration with x_{k+1}.

Here h_k is the Polyak's step in the transformed space of variables.

If B is an identity matrix, the method B turns into the method A.

Decrease of the distance to the minimum point

Theorem 3 (Stetsyuk, Stovba, Chernousova, 2018)

The sequence $\{x_k\}_{k=0}^{k^*-1}$*, generated by the method B, satisfies the inequalities*

$$\|A(x_{k+1}-x^\star)\|^2 \leq \|A(x_k-x^\star)\|^2 - \frac{m^2(f(x_k)-f^\star)^2}{\|B^T g_f(x_k)\|^2}, \ k=0,1,\ldots \quad (6)$$

Remark. Theorem 3 guarantees that in the method B the distance to the minimum point decreases monotonically in the trasformed space if the inequality (4) is used.

Method B for $f_1(x_1, x_2) = |x_1| + 10|x_2|$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1/\alpha \end{pmatrix}, \; x_0 = (1, 1)^T$$

ε_f	$\alpha = 1$	$\alpha = 1.5$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$	$\alpha = 5$
1.0e-02	262	114	62	19	17	13
1.0e-04	492	216	119	44	31	22
1.0e-06	**722**	**319**	**177**	**70**	**45**	**31**
1.0e-08	952	421	234	95	60	40
1.0e-10	1183	523	292	121	74	49

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Functions used to compare the methods A and B

1. Piecewise quadratic function

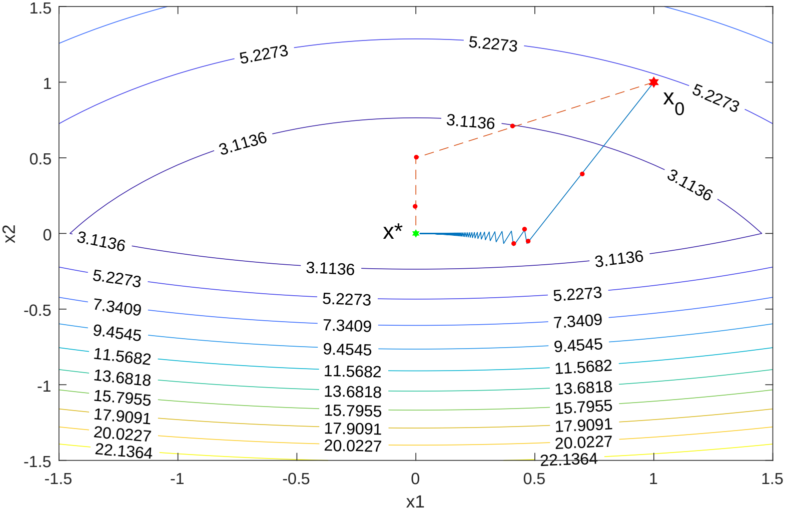
$$f_3(x_1, x_2) = \max\left\{x_1^2 + (2x_2 - 2)^2 - 3, x_1^2 + (x_2 + 1)^2\right\},$$
$$x^* = (0, 0),\ f^* = 1.$$

2. Quadratic function

$$f_4(x) = \|Ax - b\|^2,\ x^* = (1, 1, ..., 1)^T,\ f^* = 0.$$

Methods A and B for piecewise quadratic function

$x_0 = (1,1)^T$, $B = diag(1; 0.5)$, method A - blue, method B - red



Methods A and B for function $f_4(x) = \|Ax - b\|^2$

$$A = \|a_{ij}\|_{i,j=1}^{l,n} = \left\| \begin{array}{ccc} 100 & 0 & 0\ldots0 \\ 0 & 100 & 0\ldots0 \\ & A_1 & \end{array} \right\|, \quad b_i = \sum_{j=1}^{n} a_{ij},\, i = \overline{1,l},$$

A_1 - 498×100 random matrix, $m = 2$.

ε	Method A		Method B	
	itnA	$\|x_\varepsilon^* - x^*\|$	**itnB**	$\|x_\varepsilon^* - x^*\|$
1.0e-04	695	8.0612e-04	86	1.2175e-04
1.0e-06	1085	8.2861e-05	108	1.1773e-05
1.0e-08	1491	8.3470e-06	130	1.1353e-06
1.0e-10	**1901**	**8.3881e-07**	**150**	**1.3526e-07**
1.0e-12	2313	8.3994e-08	172	1.3018e-08
1.0e-14	2725	8.4440e-09	194	1.2523e-09

Conclusions

1. Using parameter $m \geq 1$ in Polyak's subgradient method gives a possibility to minimize special classes of convex functions more effectively.

2. Application of space transformation can accelerate convergence of subgradient method with Polyak's step in the transformed space of variables.

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Questions?

THANK YOU
FOR YOUR ATTENTION

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