The maximum singular value of the matrix and its economic interpretation

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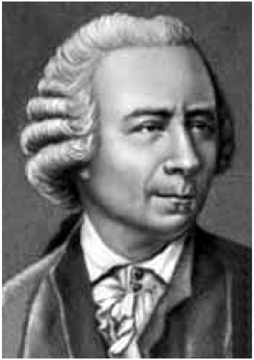

„Действительно, так как здание всего мира совершенно и возведено премудрым творцом, то в мире не происходит ничего, в чем не был бы виден смысл какого-нибудь **максимума или минимума** ...“

Л.Эйлер „Об упругих кривых“.

Abstract

We prove that the **maximum** singular value of the matrix and corresponding singular vectors are the optimal solution for the special quadratic extremal problem.

We consider the economic interpretation of the optimal solution for the linear model of production and for the productive Leontief model.

We show a connection of the optimal solution with the Frobenius number and vectors.

Comparisons of Frobenius numbers and **maximum** singular values for Leontief inverse matrix in the 15-sectoral balance of Ukraine for 2003-2009 are given.

Outline

1. Singular value σ_A and quadratic problem
2. Singular value σ_A and non-negative matrices
3. Economic interpretation of σ_A and (u^*, x^*)
4. Leontief's productive model

Singular value σ_A

Let A be real $m \times n$-matrix,
$\sigma_A > 0$ is its maximum singular value,

$$\sigma = \sqrt{\lambda_{max}(AA^T)} = \sqrt{\lambda_{max}(A^TA)},$$

where $\lambda_{max}(AA^T)$ and $\lambda_{max}(A^TA)$ are maximum eigenvalues for matrices AA^T and A^TA.

σ_A и quadratic extremum problem

Lemma 1a [1]

Let A be real $n \times m$ matrix. Its maximum singular value σ_A equals objective function optimum value in quadratic extremum problem.

$$\sigma_A = (u^*)^T A x^* = \max_{x \in R^n, u \in R^m} u^T A x \qquad (1)$$

subject to

$$\sum_{i=1}^{m} u_i^2 = 1, \quad \sum_{i=1}^{n} x_i^2 = 1. \qquad (2)$$

Here (u^*, x^*) is optimal solution for problem (1)–(2).

Non-uniqueness of solution (u^*, x^*)

Lemma 1b [1]

If σ_A multiplicity is more than 1, then optimal solution in problem (1)–(2) is either vectors

$$u^* = \xi(AA^T), \quad x^* = A^T u^* / \|A^T u^*\|, \tag{3}$$

or vectors

$$x^* = \xi(A^T A), \quad u^* = Ax^* / \|Ax^*\|, \tag{4}$$

where $\xi(AA^T)$ and $\xi(A^T A)$ are eigenvectors of AA^T and $A^T A$ matrices, corresponding to their maximum eigenvalues $\lambda_{max}(AA^T) = \lambda_{max}(A^T A)$.

Uniqueness of solution (u^*, x^*)

Lemma 2 [1]

If σ_A multiplicity equals 1, then problem (1)–(2) has a single optimal solution

$$u^* = \xi(AA^T), \quad x^* = \xi(A^TA), \tag{5}$$

where $\xi(AA^T)$ and $\xi(A^TA)$ are eigenvectors of AA^T and A^TA matrices, corresponding to their maximum eigenvalues $\lambda_{max}(AA^T) = \lambda_{max}(A^TA)$.

Frobenius numbers and vectors

Frobenius number λ_A equals

maximum eigenvalue of $n \times n$-matrix A
with non-negative coefficients.

Right Frobenius vector equals vector x_A, which is subject to

$$A x_A = \lambda_A x_A \quad \text{and} \quad \sum_{i=1}^{n} (x_A)_i = 1. \qquad \text{(frobenius1)}$$

Left Frobenius vector equals vector p_A, which is subject to

$$A^T p_A = \lambda_A p_A \quad \text{and} \quad \sum_{i=1}^{n} (p_A)_i = 1. \qquad \text{(frobenius2)}$$

Square matrix indecomposability

Matrix A called indecomposable, if by simultaneous permutation of rows and columns it cannot be transformed to a matrix looking like

$$A = \left\{ \begin{array}{cc} A_1 & A_2 \\ 0 & A_3 \end{array} \right\},$$

A_1 and A_3 are square sub-matrices with dimensions $k \times k$ and $(n-k) \times (n-k)$, respectively.

By Leontief, indecomposability of matrix A means that every sector uses products of every other sector.

Uniqueness and positiveness of solution (u^*, x^*)

Lemma 3 [1]

If non-negative $m \times n$-matrix A has no zero-filled rows and columns, and minimum by dimension from matrices AA^T and $A^T A$ is indecomposable, then singular value σ_A in problem (1)-(2) is achieved in single point (u^*, x^*), which has all components positive. Vector u^* is equal to normalized Frobenius vector for AA^T matrix, and vector x^* is equal to normalized Frobenius vector for $A^T A$ matrix.

This case takes place while analysis of economic models, where matrix coefficients are non-negative.

Linear production model $y = Ax$

Find

$$\sigma_A = (u^*)^T y^* = \max_{y \in R^m, u \in R^m} u^T y \qquad (1')$$

subject to

$$y = Ax, \quad x \in R^n, \qquad (2a')$$

$$\sum_{i=1}^{m} u_i^2 = 1, \quad \sum_{i=1}^{n} x_i^2 = 1. \qquad (2b')$$

Here (u^*, x^*) are optimal normalized vector for prices and technology use.

Linear price model $p = A^T u$

Find

$$\sigma_A = (p^*)^T x^* = \max_{p \in R^n, x \in R^m} p^T x \tag{1''}$$

subject to

$$p = A^T u, \quad u \in R^m, \tag{2a''}$$

$$\sum_{i=1}^{m} u_i^2 = 1, \quad \sum_{i=1}^{n} x_i^2 = 1. \tag{2b''}$$

Leontief's productive model $(n = m)$

Leontief models $y = (I - A)x$ and $w = (I - A^T)p$ correspond to the problem:

find

$$\sigma_B = (w^*)^T B y^* = \max_{w \in R^n, y \in R^n} w^T B y \qquad (6)$$

subject to

$$\sum_{i=1}^{n} w_i^2 = 1, \quad \sum_{i=1}^{n} y_i^2 = 1, \qquad (7)$$

where $B = (I - A)^{-1}$.

Solution of the problem (6)–(7)

(w^*, y^*) are optimum normalized vectors for final product and added value, corresponding to national income maximization [2].

For this, σ_B will be better than Frobenius number for matrix B. How much better? We'll see below.

15 sectors in Leontief's matrix (Ukraine)

матрица Леонтьева

№	Название отрасли / № отрасли	1	2	3	4	5
1	Сельское хозяйство, охотничье и лесное хозяйство	0,25644	0,07763	0,00214	0,03313	0,00017
2	Рыбное хозяйство	0,00017	0,07457	0,00001	0,00049	0,00001
3	Добывающая промышленность	0,01008	0,00367	0,06446	0,11893	0,33387
4	Перерабатывающая промышленность	0,18065	0,18032	0,15941	0,29734	0,11449
5	Производство и распределение электроэнергии, газа и воды	0,01163	0,02934	0,08042	0,02750	0,07150
6	Строительство	0,00019	0,00000	0,00107	0,00025	0,00176
7	Торговля, ремонт автомобилей, бытовых изделий и предметов личного пользования	0,12371	0,21394	0,06930	0,20672	0,00137
8	Деятельность гостиниц и ресторанов	0,00025	0,00122	0,00194	0,00116	0,00254
9	Деятельность транспорта и связи	0,04232	0,08924	0,12953	0,04982	0,01114
10	Финансовая деятельность	0,00225	0,00428	0,00749	0,00809	0,01582
11	Операции с недвижимым имуществом, аренда, инжиниринг и предоставление услуг	0,00860	0,01284	0,01234	0,01477	0,01549
12	Государственное управление	0,00032	0,00122	0,00207	0,00242	0,00726
13	Образование	0,00006	0,00000	0,00042	0,00011	0,00052
14	Здравоохранение и предоставление соц. помощи	0,00032	0,00244	0,00134	0,00045	0,00093
15	Предоставление коммунальных и индивидуальных услуг, деятельность в сфере культуры и спорта	0,00018	0,00061	0,00151	0,00083	0,00222

Fragment of 15-sector Leontief's matrix for 2009[3].

Frobenius numbers and σ_B (Ukraine, 15 sectors)

Year	λ_A	λ_B	σ_B	$\frac{(\sigma_B - \lambda_B)}{\lambda_B}$
2003	0.58641	2.41787	2.914	0.205
2004	0.58476	2.40825	2.937	0.220
2005	0.59611	2.47591	3.107	0.255
2006	0.58495	2.40936	2.980	0.237
2007	0.57231	2.33812	2.865	0.225
2008	0.56623	2.30535	2.884	0.251
2009	0.56958	2.32332	2.866	0.234

Here λ_A is Frobenius number for technology matrix A,
λ_B is Frobenius number for full expense matrix $B=(I-A)^{-1}$.

$$\lambda_B = \frac{1}{1 - \lambda_A}$$

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Запитання?

ДЯКУЮ ЗА УВАГУ!

BACKUP SLIDES: Quadratic Problem from [2]

find

$$f^* = \max_{y \in R^n, p \in R^n} p^T y \quad \equiv \quad \max_{x \in R^n, w \in R^n} w^T x \tag{1.1}$$

subject to

$$y = (I - A)x, \qquad x \geq 0, \qquad y \geq 0, \tag{1.2}$$

$$w = (I - A^T)p, \qquad p \geq 0, \qquad w \geq 0, \tag{1.3}$$

$$\|y\|^2 = 1, \qquad \|w\|^2 = 1. \tag{1.4}$$

where the $n \times n$-matrix A is known and the n-dimensional vectors x, y, p, w are unknown.

BACKUP SLIDES: Algorithm for productive A

If the matrix A is productive, then the problem (1.1)–(1.4) can be formulated as

find

$$f^* = (w^*)^T B y^* = \max_{y \geq 0,\ w \geq 0} w^T B y \tag{1.5}$$

subject to

$$\sum_{i=1}^{n} y_i^2 = 1, \qquad \sum_{i=1}^{n} w_i^2 = 1, \tag{1.6}$$

where $B = (I - A)^{-1}$.