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“Distorted"nonlinear programming Benchmarks **–** difficult ("deformed") test optimization problems and a comparative analysis of software tools

A family of test problems based on a set of basic problems is generated.This family depends on certain parameters which determine the conditioning of optimization problems. Basic problems are as follows: find

$$f^* = \min f_0(x) \tag{1}$$

subject to

$$f_k(x) \le 0, \ k = 1, \dots, m \tag{2}$$

where: $x = (x_1, x_2, \dots, x_n)$.

Test problems are generated by replacing constraints (2) by

$$\varphi_k^0(x)\varphi_k^1(x) f_k(x) \le 0, \ k = 1, \dots, m \tag{3}$$

where $\varphi_k^0(x) > 0, \ \varphi_k^1(x) > 0, \ k = 1, \dots, m$ are defined as

$$\varphi_k^0(x) = \begin{cases} \left(\mu + \left\|x - x^*\right\|^2\right)^\gamma, k = 1, \dots, m_1 \\ 1, \ k = m_1 + 1, \dots, m \end{cases},$$

$$\varphi_k^1(x) = \alpha + \sin\left(\beta \Big/ \left(\mu + (x_k - x_k^*)^2\right)\right), k = 1, \dots, m, \ \alpha > 1, \ \mu > 0.$$

Here $m_1 = [m/2]$, x^*- optimal solution of basic problems, x_k, x_k^*- k-th components of vectors x, x^*.Notice that the feasible sets of the problems are unchanged by this transformation.

The objective functions are replaced by

$$\tilde{f}_0(x) = \begin{cases} f_0(x), \text{ if } f_k(x) \le \varepsilon : k = 1, \dots, m, \\ \sqrt{-\max\{f_k(x) : k = 1, \dots, m\}}, \text{ else.} \end{cases} \tag{4}$$

where $\varepsilon \ge 0$. In the feasible region $f_0(x)$ and $\tilde{f}_0(x)$ coincide, outside the feasible region function $\tilde{f}_0(x)$ is not defined for $\varepsilon = 0$.

Note that when $\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\varepsilon = \infty$ test problems are identical to original problems (1), (2), for $\alpha = 1$, $\beta = 0$, $\gamma \ne 0$, $\varepsilon = \infty$ test problems are becoming degenerate at the optimal solution, for $\alpha > 1$, $\beta \ne 0$ and small value of μ penalty functions of test problems are multiextremal.

Problem 1. Find

$$f^* = \min\ (c, x) \tag{5}$$

subject to

$$x_i - \frac{\chi}{n}\sum_{j=1}^{n} x_j \le 0,\ i = 1, ..., n \tag{6}$$

where $\chi > 1$, $c_i = \frac{i}{10} > 0,\ i = 1, ..., n$.

Feasible set is a cone containing the vector $(1, ..., 1)$. If $\chi = 1$ then the cone degenerates into a ray generated by the vector $(1, ..., 1)$. Base point is $\mathrm{x}^0 = (1, ..., 1)$.

Solution – $x^* = 0,\ f^* = 0$.

Problem 2. Find

$$f^* = \min\left\{ f^0(x):\ x \in R^n \right\} \tag{7}$$

where $f^0(x) = \max\{ f_k(x) : k = 1, ..., n\}$, $f_k(x) = \sum_{i=1}^{n} (x_i - x_i^k)^2$, $x^k = (x_1^k, ..., x_n^k)$ are given points, $k = 1, ..., n$. and $x_i^k = 0,\ if\ i \ne k$, $x_i^k = 1,\ if\ i = k$, $k = 1, ..., n,\ i = 1, ..., n$.

Equivalent formulation: find

$$\min\ Y \tag{8}$$

subject to

$$f_k(x) - Y \le 0,\ k = 1, ..., n \tag{9}$$

Starting point $\mathrm{x}^0 = (1,\ 0, ...,\ 0)$, $y^0 = 20$. Solution– $x^* = \left(\frac{1}{n}, ..., \frac{1}{n}\right),\ f^* = y^* = \frac{n-1}{n}$.

Problem 3. Find

$$f^* = \min_x \left\{ \lambda \cdot (p, x) - \sqrt{\delta^2 \cdot (p, x)^2 - \|x - (x, p) \cdot p\|^2} \right\} \tag{10}$$

subject to

$$\delta^2 \cdot (p, x)^2 \ge \|x - (x, p) \cdot p\|^2 + \sigma^2,\ x \in R^n \tag{11}$$

$$(p, x) \ge 0,\ x \in R^n \tag{12}$$

where p - given vector, $\|p\| = 1$, $p_i = 1/\sqrt{n}$, $i = 1,\dots, n$, $\lambda > \delta > 0$, $\sigma > 0$ - given numbers. If $\sigma = 0$ then norm of gradient of the objective function equals $+\infty$ at the boundary of the feasible set.

Starting point x^0 was chosen with some displacement from the ray generated by the vector p.

If x^* - optimum solution, then $x^* = pt^*$ where $t^* \in R$. Hence $(p, x) = t^*$, $\delta \cdot t^* = \sigma$. Solution is $x^* = p \cdot \frac{\sigma}{\delta}$, $f^* = \lambda \cdot \frac{\sigma}{\delta} - \sigma = \sigma(\frac{\lambda}{\delta} - 1)$.

All test problems are coded in AMPL language (http://www.ampl.com/index.html), the prepared files are ready for solution in the AMPL environment. In the attached files (file extension - mod) the test problems of minimizing the objective function of the form (4) under constraints (3) were generated from the original problems 1 - 3. The number of variables in all problems (the dimension of the vector x) is 50. These files contain commands AMPL to load different AMPL solvers for different values of parameters that define the ill-conditioned problems. Parameters are chosen so that these problems are hard to solve by standard solvers snopt, minos, loqo. A comparison was made with the nonsmooth penalty function method (solver AmplRalg) and the method of convex extensions (solver ConExp). The results of solution of the problems are recorded in files with the extension *.rez* in subdirectory *Test_Pr*, located in the directory, which contains all AMPL files (where AMPL is loaded).

A brief description of the given files:

PR_1_1.mod - *base problem 1*, $\alpha = 1.1$, $\beta = 0$, $\gamma = 0$, $\varepsilon = 1.e16$, $\mu = 0.001$. With these values of the parameters the perturbing functions $\varphi_k^0(x), \varphi_k^1(x),\ k = 1,\dots, m$ are constants ($\varphi_k^0(x) \equiv 1,\ \varphi_k^1(x) \equiv 1.1$). Problems are designed to compare the method of convex cone extensions (solver ***ConExp***) and the method of nonsmooth penalty functions (solver ***AmplRalg***) and were solved for different values χ. If the value of penalty coefficient equal to 1000 (the "default") for $\chi = 1.10$ and $\chi = 1.05$, then the penalty function is not bounded below. This is due to the necessity to choose penalty coefficient individually for each problem. Method of convex cone extensions provides slightly higher accuracy in the objective function and fewer number of function calls in r - algorithm than when using the method of nonsmooth penalty function. Every function call in this case a much more laborious procedure since the problem of one-dimensional search is solved. File of results - ***PR_1_1.rez***. The results are shown in Table 1.1.

Table 1.1.Comparison of solvers ConExp (method of convex conical extensions) and AmplRalg (method of nonsmooth penalty functions). Optimal value of the objective function - 0.

χ	**Method**	**Record value of the objective function**	**The number of function calls**
1.50	Convex extension	0.0006658	875
	Nonsmooth penalties	0.0010041	1259
1.45	Convex extension	0.0004572	922
	Nonsmooth penalties	0.0010863	1188
1.40	Convex extension	0.0015920	990
	Nonsmooth penalties	0.0026862	959
1.35	Convex extension	0.0015888	827
	Nonsmooth penalties	0.0023529	1037
1.30	Convex extension	0.0009020	994
	Nonsmooth penalties	0.0023775	1036
1.25	Convex extension	0.0011574	1098
	Nonsmooth penalties	0.0066865	1018
1.20	Convex extension	0.0011774	1096
	Nonsmooth penalties	0.0011554	1045
1.15	Convex extension	0.0068407	882
	Nonsmooth penalties	0.0027255	1044
1.10	Convex extension	0.0080652	992
	Nonsmooth penalties	F	668
1.05	Convex extension	0.0026539	1162
	Nonsmooth penalties	F	461

PR_1_2.mod - *base problem 1,* $\alpha = 1.1$, $\beta = 0$, $\varepsilon = 1.e16$, $\mu = 1\text{e-}16$, $\chi = 1.1$. With these values of the parameters perturbing functions $\varphi_k^1(x),\ k = 1,\ldots,m$ are constants ($\varphi_k^1(x) \equiv 1.1$). Functions $\varphi_k^0(x),\ k = 1,\ldots,m$ define degenerate scaling at the optimum point. Problems are designed to compare different solvers and can be solved for different values of γ. For negative γ all software tools, except ***ConExp*** (method of convex extensions**),** found incorrect solutions. The objective function value at the solution found by method of convex extensions, also depends on the value of γ. For positive γ differences in the solutions found by different solvers were less significant. Results file - ***PR_1_2.rez.*** The results are shown in Table. 1.2.

Table 1.2. Objective function values obtained by different solvers based on the value of γ. Optimal objective function value - 0.

γ	**snopt**	**minos**	**loqo**	**ConExp**	**AmplRalg**
3.	F	F	F	0.8031	F
2.5.	F	F	F	0.4629	F
2.	F	F	F	0.2354	F
1.5.	F	F	F	0.0856	F
-1	F	F	F	0.01468	F
0.5	F	F	F	0.0078	0.0199
0	0	0	0	0.0080	F
0.5	0.0104	0.00006	F	0.0017	F
1	0.2359	0.0258	F	0.0206	F
1.5.	0.8902	0.1621	F	0.0899	F
2	1.6615	0.6147	F	0.2741	F
2.5	F	11.1109	F	0.5577	F
3	3.1324	F	F	0.9132	F

PR_1_3.mod - *base problem* 1, $\alpha = 1.1$, $\beta = 0$, $\varepsilon = 0.00001$, $\mu = 1\text{e-}16$, $\chi = 1.1$. Functions $\varphi_k^0(x)$, $\varphi_k^1(x)$, $k = 1,..., m$ are the same as in file ***PR_1_2.mod***. Boundary for the region of definition of objective function was added. Problems are solved for different values of γ. File of results - ***PR_1_3.rez***. The results are shown in Table. 1.3.

Table 1.3. Objective function values obtained by different solvers based on the value of γ. Optimal objective function value - 0.

γ	**snopt**	**minos**	**loqo**	**ConExp**
0.0	0.0000	0.0000	F	0.0080
0.5	0.0104	0.00006	F	0.0017
1.0	0.2359	F	F	0.0206
1.5.	1.0088	F	F	0.0899
2.0	1.6615	F	F	0.2741
2.5	1.2714	F	2.7193	0.5577
3.0	F	F	4,646	0.9132

PR_1_4.mod - *base problem 1,* $\alpha = 1.1$, $\gamma = 0$, $\varepsilon = 1.e16$, $\mu = 1e\text{-}16$, $\chi = 1.15$. Perturbing functions $\varphi_k^0(x),\ k = 1,\ldots, m$ are constants ($\varphi_k^0(x) \equiv 1$). Functions $\varphi_k^1(x),\ k = 1,\ldots, m$ define oscillating factor for the functions of constraints**.** Problems are solved for different values of parameter β. The results of calculations also show the advantages of the method of convex extensions (solver ***ConExp).*** Results file - ***PR_1_4.rez.*** The results for β = 0,...,5 are shown in Table.1.4.

Table 1.4. Objective function values obtained by different solvers based on the value of β. Optimal objective function value - 0.

β	**snopt**	**minos**	**loqo**	**ConExp**	**AmplRalg**
0.	0	0	0	0.0068	0.0027
1	F	F	F	0.0816	F
2	F	F	F	0.1210	F
3	F	F	F	0.1326	F
4	F	F	F	0.1714	F
5	36.74	F	F	0.1497	F

PR_2_1.mod - *base problem 2,* $\alpha = 1.1$, $\beta = 0$, $\gamma = 0$, $\varepsilon = 1.e16$, $\mu = 0.001$. With these values of parameters the perturbing functions $\varphi_k^0(x), \varphi_k^1(x), k = 1,\ldots, m$ are constants ($\varphi_k^0(x) \equiv 1,\ \varphi_k^1(x) \equiv 1.1$). ***snopt, minos, loqo*** solvers solved the problems successfully. File of results - ***PR_2_1.rez***. ***ConExp*** (the method of convex extensions) also solves the problem successfully - found the objective function value is equal to 0.9800059 (exact value - 0.98), the number of calls of the procedure computing the objective function - 938. Behavior of AmplRalg (the method of nonsmooth penalty functions) depends strongly on the penalty coefficients. This dependence is shown in Table. 2.1.

Table 2.1. Behavior of AmplRalg

Penalty coefficients	**Record value of the objective function**	**The number of function calls**
1000	1.05748	861
100	0.983084	5448
10	0.980009	3108
1	0.980006	786

PR_2_2.mod - *base problem 2,* $\alpha = 1.1$, $\beta = 0$, $\varepsilon = 1.e16$, $\mu = 1e\text{-}16$, $\chi = 1.1$. With these values of parameters the perturbing functions $\varphi_k^1(x),\ k = 1,\ldots, m$ are constants ($\varphi_k^1(x) \equiv 1.1$).

Functions $\varphi_k^0(x),\ k=1,...,m$ define a degenerate scaling in the optimum point. Results file - ***PR_2_2.rez***. The results are shown in Table. 2. Penalty coefficient (for ***AmplRalg***) - 1000.

Table 2.2. Objective function values obtained by different solvers based on the value of γ (optimal value - 0.98)

γ	**snopt**	**minos**	**loqo**	**ConExp**	**AmplRalg**
2.5.	1.0902.	17.6651.	2.0093	0.9954	1.3953
2.	0.9876	15.4554.	1.0455	0.9883	1.3355
1.5.	1.0017	13.2207	6.0260	0.9829	1.2548
-1	0.9800	12.1226	3.9188	0.9806	1.1090
0.5	0.9800	0.9800	0.9800	0.9800	1.0460
0	0.9800	0.9800	0.9800	0.9800	1.0574
0.5	0.9799	12.9437	0.9799	0.9800	0.9827
1	0.9796	8.6508	0.9796	0.9801	F
1.5.	F	F	0.9774	0.9809	1.0117
2	F	F	F	0.9832	1.0069
2.5	F	F	F	0.9909	F
3	F	F	F	0.9952	F

PR_2_4.mod - *base problem 2,* $\alpha=1.1$, $\gamma=0$, $\varepsilon=1.e16$, $\mu=1\text{e-}16$. Perturbing functions $\varphi_k^0(x),\ k=1,...,m$ are constants ($\varphi_k^0(x)\equiv 1$). Functions $\varphi_k^1(x),\ k=1,...,m$ set oscillating factor for the functions of constraints. Problems are solved for different values of parameter β. Results of calculations also show the advantages of the method of convex extensions (solver ***ConExp***). Results file - ***PR_2_4.rez***. The calculation results for β = 0, ..., 5 are shown in Table. 2.4. Penalty coefficient (for ***AmplRalg***) - 1000.

Table 2.4. Objective function values obtained by different solvers based on the value of β (optimal value - 0.98)

β	***snopt***	***minos***	***loqo***	***ConExp***	***AmplRalg***
0.	0.9800	0.9800	0.9800	0.9800	1.0574
1	1.0340	6.6412	1.5861	0.9803	1.5085
2	F	F	F	0.9805	1.0099
3	F	20.0000	1.6841	0.9808	6.0519
4	F	8.2105	1.4978	0.9810	F
5	F	F	1.6464	0.9809	2.5780

PR_3_1.mod - *base problem 3*. The problem was specifically chosen as a difficult problem for the method of convex extensions (the norm of the gradient of the objective function equals $+\infty$ at the boundary of the feasible set for $\sigma = 0$). Base problem ($\varphi_k^0(x) \equiv 1,\ \varphi_k^1(x) \equiv 1,\ k = 1,\dots,m$, $\tilde{f}_0(x) \equiv f_0(x)$) is solved directly with $\lambda = 0.02$ and different values of the parameters δ, σ. The results of calculations also show that the method of convex extensions (solver ***ConExp***) is more stable for ill-conditioned problems. Results file - ***PR_3_1.rez***. For solver ***minos*** program errors appeared. For this fact solver ***minos*** was excluded from the computational experiments. When comparing the results of calculations the resulting value of the objective function and the degree of constraint violations should be considered.

PR_3_2.mod - *base problem 3*. Perturbing functions $\varphi_k^0(x) \equiv 1,\ \varphi_k^1(x) \equiv 1,\ k = 1,\dots,m$, are constants. Boundary for the definition region of objective function was added (using the objective function of the form (4), $\varepsilon = 1\text{e-}9$). The problem is solved with $\lambda = 0.02$ and different values of the parameters δ, σ. The results of calculations are presented in Table. 3.2. Results file - ***PR_3_2.rez***. Method of convex extensions (solver ***ConExp***) is more stable.

Table 3.2.Objective function values obtained by different solvers based on values of δ and σ.

σ	**Optimal value**	**snopt**	**loqo**	**ConExp**
δ = 0. 00003				
0. 0010	0.66567	0.9847087	94.3125	0.66567
0. 0008	0.53253	0.533591	73.7437	0.53253
0.0006	0.39940	0.407735	56.6526	0.39940
0. 0004	0.26627	0.279845	37.3732	0.26627
0.0002	0.13313	0.157500	16.4972	0.13313
δ = 0. 00002				
0.0010	0.99900	1.01032	141,701	0.99900
0.0008	0.79920	0.80177	112,036	0.79920
0.0006	0.59940	0.69740	84,389	0.59940
0.0004	0.39960	0.40417	54,478	0.39960
0.0002	0.19980	0.26016	27,622	0.19980